Permutations and Expansion:

<u>Defn</u>: Let X be a set.
A bijection $f: X \longrightarrow X$ is called a "permutation" of X.

<u>Notation</u>: Let $Aut(X)$ denote the set of permutations of X

We will mainly be interested in pertutations of the set $X = \{1, \dots, n\}$.

<u>For example</u>: Let $X = \{1, 2, 3\}$

Define $f \in Aut(X)$ by
$$f(1) = 2$$
$$f(2) = 1$$
$$f(3) = 3$$

<u>Notation</u>:
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & & n \\ f(1) & f(2) & f(3) & & f(n) \end{bmatrix}$$

<u>Properties</u>

(1) $f, g \in Aut(X) \Longrightarrow f \circ g \in Aut(X)$
(2) $f \in Aut(X) \Rightarrow f^{-1}$ exists and $f^{-1} \in Aut(X)$
(3) $id \in Aut(X)$ and $id \circ f = f = f \circ id \;\; \forall f \in Aut(X)$

$Aut(X)$ is a "group"

$$g = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \rightsquigarrow f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$f^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Definition: $f \in Aut(X)$ is called a transposition iff there exist $x, x' \in X$ $x \neq x'$ such that

(i) $f(x) = x'$

(ii) $f(x') = x$, and

(iii) $f(y) = y \quad \forall\, y \notin \{x, x'\}$

Theorem: Let X be a finite set, $\# X \geq 2$, Then for each $f \in Aut(X)$ there exist transpositions $\tau_1, \ldots, \tau_k \in Aut(X)$ so that

$$f = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k$$

Proof: By induction on the number of elements in the set X.

Base case: $\#X = 2$. $X = \{x, x'\}$

Then $Aut(X) = \{id, \tau\}$ $\quad \tau(x) = x'$, $\tau(x') = x$

$id = \tau \circ \tau$

$\tau = \tau$

Inductive step:

Suppose true for all X' with $\#X' = n$

Wish to prove true for X with $\#X = n+1$.

Given X with $\#X = n+1$.

Let $f \in Aut(X)$

If $f = id$, then $f = \tau \circ \tau$

Otherwise, $\exists\, x \in X$ so that $f(x) \neq x$

Let τ be transposition associated to x and $f(x)$

Observe that $\tau(f(x)) = x$

Therefore we can restrict $(\tau \circ f)$ to the set $X \setminus \{x\}$ and obtain a well-defined function

$$\tau \circ f\big|_{X \setminus \{x\}} \quad X \setminus \{x\} \longrightarrow X \setminus \{x\}$$

Since $\tau \circ f$ is a bijection this map is a bijection.

Since $^{\#}(X \setminus \{x\}) = n$, the inductive hypothesis gives that there exist transpositions of $X \setminus \{x\}$ $\tau_1, \ldots, \tau_k$ so that

$$(\tau \circ f)\Big|_{X \setminus \{x\}} = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k.$$

Extend these to transpositions of X by setting

$$\tilde{\tau}_i(y) = \tau_i(y) \quad \forall\, y \in X \setminus \{x\}$$
$$\tilde{\tau}_i(x) = x$$

Note that $(\tau \circ f)(x) = x = \tilde{\tau}_1 \circ \tilde{\tau}_2 \cdots \circ \tilde{\tau}_k(x)$ and hence

$$\tau \circ f = \tilde{\tau}_1 \circ \tilde{\tau}_2 \circ \cdots \circ \tilde{\tau}_k$$

Thus $f = \tau \circ \tilde{\tau}_1 \circ \tilde{\tau}_2 \circ \cdots \circ \tilde{\tau}_k$.

This completes the proof of the inductive step.

<u>Example</u>:

The sign of a permutation

The factorization of a permutation into transpositions is not unique.
For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \circ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

and $$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \circ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

Let X be a finite set.
Let $\varphi : X \longrightarrow \{1, 2, \dots, n\}$ be a bijection

Let $f \in Aut(X)$. Define $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g(x_1, \dots, x_n) = \frac{\prod_{i<j} \left(x_{\varphi \circ f \circ \varphi^{-1}(i)} - x_{\varphi \circ f \circ \varphi^{-1}(j)} \right)}{\prod_{i<j} (x_i - x_j}$$

<u>Example</u>: $X = \{ \text{red}, \text{blue}, \text{green} \}$

$$\varphi = \begin{bmatrix} \text{red}, & \text{blue}, & \text{green} \\ 1, & 2, & 3 \end{bmatrix}$$

$$f = \begin{bmatrix} \text{red} & \text{blue} & \text{green} \\ \text{blue} & \text{green} & \text{red} \end{bmatrix}$$

$$\varphi \circ f \circ \varphi^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$g(x_1, \dots, x_n) = \frac{\cancel{(x_2 - x_3)}(x_2 - x_1)(x_3 - x_1)}{(x_1 - x_2)(x_1 - x_3)\cancel{(x_2 - x_3)}}$$

$$= \frac{(x_2 - x_1)}{(x_1 - x_2)} \frac{(x_3 - x_1)}{(x_1 - x_3)} \frac{(x_2 - x_3)}{(x_2 - x_3)}$$

$$= (-1) \cdot (-1) \cdot (+1)$$

$$= +1$$

In general, each factor in numerator matches a factor in denominator up to a sign. Therefore g is a constant function and

$$g \equiv 1 \quad \text{or} \quad g \equiv -1.$$

Proposition:

The value of g depends only $f \in Aut(X)$ and not on the bijection φ.

Pf: Let $\bar{\varphi} : X \longrightarrow \{1, 2, \dots, n\}$ be another bijection

$$\bar{g}(x_1, \dots, x_n) = \frac{\prod_{i<j} \left(x_{\bar{\varphi} \circ f \circ \bar{\varphi}^{-1}(i)} - x_{\bar{\varphi} \circ f \circ \bar{\varphi}^{-1}(j)}\right)}{\prod_{i<j} (x_i - x_j)}$$

ISTS

$$\prod_{i<j} \left(x_{\bar{\varphi} \circ f \circ \bar{\varphi}^{-1}(i)} - x_{\bar{\varphi} \circ f \circ \bar{\varphi}^{-1}(j)}\right)$$

$$\prod_{i<j} \left(x_{\varphi \circ f \circ \varphi^{-1}(i)} - x_{\varphi \circ f \circ \varphi^{-1}(j)}\right)$$

Definition: The value of g is called the sign of the permutation f.
Let $\varepsilon(f) =$ sign of permutation.

Exercise: sign does not depend on φ

Proposition: If τ is the transposition associated to k and l, then $\varepsilon(\tau) = -1$

Pf: Suppose $k < l$. Then

$$g_\tau(x_1, \dots, x_n) = \frac{\prod_{i<j} (x_{\tau(i)} - x_{\tau(j)})}{\prod_{i<j} (x_i - x_j)} = \frac{(x_l - x_k)}{(x_k - x_l)} = -1.$$

If $k > l$, then $g_\tau(x_1, \dots, x_n) = \dfrac{(x_k - x_l)}{(x_l - x_k)} = -1.$ $\blacksquare$

Proposition: $f, g \in \mathrm{Aut}(X) \Rightarrow \varepsilon(f \circ g) = \varepsilon(f) \cdot \varepsilon(g)$

Proof:

$$\prod_{i<j} (x_{f\circ g(i)} - x_{f \circ g(j)})$$

$$\left.\begin{aligned} &= \prod_{i<j} (x_{f(g(i))} - x_{f(g(j))}) \\ &= \varepsilon(g) \prod_{i<j} (x_{f(i)} - x_{f(j)}) \end{aligned}\right\} \text{EXERCISE!}$$

$$= \varepsilon(g) \cdot \varepsilon(f) \prod_{i<j} (x_i - x_j)$$

Corollary: Let $f = \tau_1 \circ \cdots \circ \tau_k$ be a factorization into transpositions. Then $\epsilon(f) = (-1)^k$.

Pf:
$$\begin{aligned} \varepsilon(f) &= \varepsilon(\tau_1 \circ \tau_2 \circ \cdots \circ \tau_k) \\ &= \varepsilon(\tau_1) \cdot \varepsilon(\tau_2) \cdots \varepsilon(\tau_k) \\ &= (-1) \cdot (-1) \cdots (-1) \\ &= (-1)^k \end{aligned}$$

Corollary: Given $f \in Aut(X)$, then the parity of the number of factorizations of f into transpositions is constant. That is, Each factorization has either an even number of factors or an odd number of factors.

Application to alternating multilinear form

Recall property $\underline{III}$ of $D : \underbrace{K^n \times K^n \times \cdots K^n}_{n \text{ times}} \longrightarrow K$

$$D(v_1, v_2, \ldots, v_j, v_{j+1}, \ldots v_n) = - D(v_1, \ldots, v_{j+1}, v_j, \ldots, v_n)$$

Rephrased: Let τ be the transposition associated to $i \& j$: $\tau(j) = j+1$, $\tau(j+1) = j$ and $\tau(i) =$ $\forall i \neq j$ or $j+1$
Then $\underline{III}$ can be written as

$$(*) \quad D(v_1, v_2, \ldots, v_n) = - D(v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n)})$$

We will call a transposition "successive" if $\tau(j) = \tau(j+1)$ and $\tau(j+1) = j$.

We wish to show that (*) holds for any transposition. To this end we prove:

<u>Lemma</u>: Let $f \in \mathrm{Aut}(\{1, \dots, n\})$ be a transposition. Then $f = \tau_1 \circ \cdots \circ \tau_k$ where each τ_i is a successive transposition.

<u>Pf</u>: For the transposition f associated to k and ℓ either $k < \ell$ or $\ell < k$. By relabeling k and ℓ if needed, we may assume $k < \ell$. Thus $f(k) = \ell > k$.

We will prove the Lemma by induction on the "distance" $f(k) - k$.

<u>Base case</u>: $f(k) - k = 1$.

Then f is successive and we are done

<u>Inductive step</u>:

Suppose true $\forall$ transpositions f' w/ $f'(k) - k = j$

We want to show true for transpositions f with $f(k) - k = j+1$.

So let f be transposition w/ $f(k) - k = j+1 \geq 2$

Let τ = transposition associated to $f(k)$ and $f(k) - 1$.

Then since $f(k) - k \geq 2$, $f(k) - 1 > k$.

In particular, $\tau \circ f$ is the transposition associated to k and $f(k) - 1$

Thus $(\tau \circ f)(k) - k = f(k) - 1 - k = j$
and so by the inductive hypothesis we have

$$\tau \circ f = \tau_1 \circ \cdots \circ \tau_k$$

where each τ_i is a successive transposition.
Hence

$$f = \tau \circ \tau_1 \circ \cdots \circ \tau_k$$

and so since τ is also a successive transposition, f is factorized by successive transpositions

Now given any transposition $f \in Aut(\{1, \dots, n\})$ factorize it into successive transpositions

$$f = \tau_1 \circ \cdots \circ \tau_k$$

$$D(v_{f(1)}, \dots, v_{f(n)}) = D(v_{\tau_1 \circ \cdots \circ \tau_k(1)}, \dots, v_{\tau_1 \circ \cdots \circ \tau_k(n)})$$

$$= (-1) D(v_{\tau_2 \circ \cdots \tau_k(1)}, \dots, v_{\tau_2 \circ \cdots \circ \tau_k(n)})$$

$$= (-1)^2 D(v_{\tau_3 \circ \cdots \circ \tau_k(1)}, \dots, v_{\tau_3 \circ \cdots \tau_k(n)})$$

$$\vdots$$

$$= (-1)^k D(v_1, \dots, v_n)$$

$$= \varepsilon(f) D(v_1, \dots, v_n)$$
$$= -D(v_1, \dots, v_n)$$

The same computation gives:

Proposition: For any permutation $f \in Aut(\{1,\dots,n\})$ we have $D(v_{f(1)},\dots,v_{f(n)}) = \varepsilon(f) D(v_1,\dots,v_n)$

This formula allows us to express how the value of D changes when one n-tuple $v_1,\dots,v_n$ of vectors is replaced with another n-tuple of vectors $w_1,\dots,w_n$. In particular, suppose there exist $a_{ij} \in K$ so that

$$w_1 = a_{11} v_1 + \cdots + a_{n1} v_n$$
$$w_2 = a_{12} v_1 + \cdots + a_{n2} v_n$$
$$w_n = a_{1n} v_1 + \cdots + a_{nn} v_n$$

Lemma 7.1: (Expansion formula)

$$D(w_1,\dots,w_n) = \sum_{f \in Aut(\{1,\dots n\})} \varepsilon(f)\, a_{f(1)1} a_{f(2)2} \cdots a_{f(n)n} D(v_1,\dots,v_n)$$

Proof: Let $End(\{1,\dots,n\})$ denote the set of all maps $f: \{1,\dots,n\} \longrightarrow \{1,\dots,n\}$ including those that are not bijective.

Apply multilinearity of D argument by argument. We obtain a sum whose terms are indexed by the choice of n-tuple of numbers from $\{1, \dots, n\}$. There is a 1-1 correspondence between these n-tuples and elements of $End(\{1, \dots, n\})$. In particular, the n-tuple can be uniquely written as $(f(1), f(2), \dots, f(n))$ where $f \in End(\{1, \dots, n\})$.

This reasoning leads to

$$D(w_1, \dots, w_n) = \sum_{f \in End(\{1, \dots, n\})} D(a_{f(1)1} v_{f(1)}, \dots, a_{f(n)n} v_{f(n)})$$

$$= \sum_{f \in End(\{1, \dots, n\})} a_{f(1)1} \cdots a_{f(n)n} D(v_{f(1)}, \dots, v_{f(n)})$$

Note that if f is not a bijection then

$$D(v_{f(1)}, \dots, v_{f(n)}) = 0.$$

Therefore, the last sum equals

$$= \sum_{f \in Aut(\{1, \dots, n\})} a_{f(1)1} \cdots a_{f(n)n} D(v_{f(1)}, \dots, v_{f(n)})$$

Now we can apply the preding proposition to obtain

$$= \sum_{f \in Aut(\{1,\dots,n\})} a_{f(1)1} \cdots a_{f(n)n} \cdot \varepsilon(f) \cdot D(v_1, \dots, v_n)$$
