M413 - HW10 - Enrique Areyan - Fall 2013 ①

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Pages 99-100:

(8) Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$.
Prove that f is bounded on E. Show that the conclusion is false if
boundedness of E is omitted from the hypothesis.

Pf: that $f(E)$ is bounded:

We have that:

ⓐ f is uniformly continuous on E:
$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x,y \in E :$ If $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$

ⓑ E is bounded.
$\exists M \in \mathbb{R} : \forall x \in E : |x| < M$. Equivalently, $\exists r > 0$ and $p \in \mathbb{R}$ s.t. $f(E) \subset N_r(p)$.

We want to prove that $f(E)$ is bounded. Suppose $f(E)$ is not bounded

ⓒ $\forall M' \in \mathbb{R} : \exists x \in E : |f(x)| > M'$.

Let $\varepsilon = M = M'$. Pick δ s.t. ⓐ holds. Pick $x \in E$ s.t. ⓒ holds. Then:
If $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$. But E is bounded, so by ⓑ
we have $|x-y| < \delta < M$, since the difference between two numbers on E
cannot exceed their bound. But now we have:

$$\begin{array}{l} -M < x-y < M \\ -M < f(x)-f(y) < M \\ \hline 0 < f(x)-f(y)-(x-y) < 0 \end{array}$$ substracting the two equations:

And so we have found a number that is bigger and smaller than zero
at the same time. A clear contradiction. ✓

Now, the conclusion is false if we remove boundedness of E, for
consider $f(x) = x$ a uniformly continuous function, and $E = (0,\infty)$
then clearly $f(E) = f((0,\infty)) = (0,\infty)$, which is not bounded.
Therefore, the result holds true provided that E is bounded.

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(11) Suppose f is a uniformly continuous mapping of a metric space X
into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence
in Y for every Cauchy sequence $\{x_n\}$ in X. ✓
Use this result to give an alternative proof of the theorem stated
in Exercise 13.

We have that

ⓐ f is uniformly continuous on X:

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x, y \in X$: If $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

ⓑ $\{x_n\}$ is a Cauchy sequence on X:

$\forall \varepsilon > 0 : \exists N : \forall n, m \geq N : d_X(x_n, x_m) < \varepsilon$.

We want to prove that $\{f(x_n)\}$ is Cauchy in Y.

Let $\varepsilon > 0$. Pick $\delta > 0$ s.t. ⓐ holds. Now for that δ, pick N s.t. $n, m \geq N$ ⓑ holds, i.e., $d_X(x_n, x_m) < \delta \overset{\text{by (a)}}{\Longrightarrow} d_Y(f(x_n), (x_m)) < \varepsilon$. So, for $\varepsilon > 0$, pick N as before to conclude that, if $n, m \geq N$ then $d_Y(f(x_n), f(x_m)) < \varepsilon$, so $\{f(x_n)\}$ is Cauchy in Y.

Now, let us use this result to give an alternative proof of the theorem stated in Exercise 13. $(f : E \subset X \to \mathbb{R})$.

We want to prove: Let E be a dense subset of a metric space X, and let f be uniformly continuous real function defined on E. Prove that f has a continuous extension from E to X, i.e., there exists continuous real function g on $\mathbb{R}$ s.t. $g(x) = f(x)$ for all $x \in E$. $(g : X \to \mathbb{R})$.

$E \subset \mathbb{R}$, dense in X, if every point of X is a limit point of E, or a point of E (or both). So consider $x \in \bar{E}$. If $x \notin X$ then x is a limit point of X. Let us define our function g as follow: $g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} f(x) & \text{if } x \in E \\ \lim_{n \to \infty} x_n & \text{if } x \notin \bar{E} \end{cases}$$

So if $x \notin E$, x is a limit point of E.

Note that $g(x) = f(x)$ for all $x \in E$. Now we need to show that f is continuous, i.e., $\forall \varepsilon > 0 : \exists \delta > 0$ s.t. $d_X(p,q) < \delta \Rightarrow d_Y(g(p), g(q)) < \varepsilon$. Let us do this by cases: (i) If $p \in E$ let $\varepsilon > 0$, then $\exists \delta > 0$ s.t. $\forall p, q \in E$, $d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon \Leftrightarrow d_Y(g(p), g(q)) < \varepsilon$ by def of g.

(ii) If $q \in E^c$, $x_n \to q$, $g(x_n) \to g(q)$. Let $\varepsilon > 0$. Pick $\delta > 0$ s.t. $d_X(p, x_n) < \delta \Rightarrow d_Y(g(p), g(x_n)) < \varepsilon/2$ $\forall x_n \in E$. Pick N s.t. $d_Y(g(x_n), g(q)) < \frac{\varepsilon}{2}$ $n \geq N$. Pick $k \geq N$ s.t. $\forall n_k \geq k$: $d_X(x_{n_k}, q) < \delta$. then $d_X(p,q) < \delta$,

$d_Y(g(p), g(q)) \leq d_Y(g(p), g(x_{n_k})) + d_Y(g(x_{n_k}), g(q)) < \varepsilon$.

(14) Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Solution: Consider the following cases:

If $f(0) = 0$ then we are done.

If $f(1) = 1$ then we are done.

Otherwise, If $f(0) \neq 0$ and $f(1) \neq 1$, define $g(x) = f(x) - x$, $x \in [0$

Now, $g(0) = f(0) - 0 = f(0)$, but $f(0) \neq 0$ so $f(0) > 0$

$\Rightarrow 0 < g(0)$

Also, $g(1) = f(1) - 1$, but $f(1) \neq 1$ so $f(1) < 1$

$\Rightarrow g(1) < 0$,

So $g(1) < 0 < g(0)$. Since g is a continuous mapping (it is the difference of two continuous mapping $f(x)$ and x), so there exist $x \in [0,1]$ s.t. $g(x) = 0 \Leftrightarrow g(x) = f(x) - x = 0 \Rightarrow f(x) = x$.

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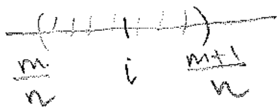
(18) Every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by:

$$f(x) = \begin{cases} 0 & (x \text{ irrational}) \\ \frac{1}{n} & (x = \frac{m}{n}) \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Pf: Schematically: , i an irrational number (fixed for the p

$i \in (\frac{m}{n}, \frac{m+1}{n})$, where

Let $\varepsilon > 0$. Pick $\frac{1}{n} < \varepsilon$. Now, find m and $m+1$ s.t. $\gcd(m,n) = \gcd(m+1,n) = 1$. that you can find such and m and $m+1$ follow from the arquimedian property of real numbers. Now, notice that for any rational number $x = \frac{a}{b}$ (again, $\gcd(a,b) = 1$) s.t. $x \in (\frac{m}{n}, \frac{m+1}{n})$ we have hat $b > n$, or otherwise $x \notin \left(\frac{m+1}{n}, \frac{m}{n}\right)$. therefore, by our definition f x we have that:

$$f(x) = \frac{1}{b} < \frac{1}{n} < \varepsilon; \text{ but Note that}$$

$$|f(x) - f(i)| = |f(x) - 0| = |f(x)| < \varepsilon$$

since f of an irrational is zero.

he other case when $y \in \left(\frac{m}{n}, \frac{m+1}{n}\right)$ is s.t. y is irrational is trivial because $|f(y) - f(i)| = |0 - 0| = 0 < \varepsilon$.

herefore, Pick $\delta = \min\left(\left|x - \frac{m+1}{n}\right|, \left|x - \frac{m}{n}\right|\right)$, and by previous gument the result follows.

ow, to prove that f has a simple discontinuity at every rational onsider the sequence $\left\{\frac{1}{n} + \frac{a}{b}\right\}$, where $\frac{a}{b}$ is an arbitrary but fixed tional number. Clearly $\left\{\frac{1}{n} + \frac{a}{b}\right\} \to \frac{a}{b}$. However,

$$f\left(\frac{1}{n} + \frac{a}{b}\right) = f\left(\frac{b+an}{bn}\right) = \begin{cases} \frac{1}{bn} & \text{if } b \nmid b+an \\ \frac{1}{n} & \text{if } b \mid b+an \end{cases}$$

t then $\lim_{n \to \infty} f\left(\frac{b+an}{n}\right) = \begin{cases} \lim \frac{1}{bn} \\ \lim \frac{1}{n} \end{cases} = 0$; So this limit is zero

ut $f\left(\frac{a}{b}\right) = \frac{1}{b} \neq 0$, and therefore, since our choiche of $\frac{a}{b}$ was bitrary, this shows, by definition of continuity by sequences, that is not continuous at every rational point. Moreover, this uence shows that $f\left(\frac{a}{b}^+\right)$ exists (appraching from the right). A very ilar argument, but using the sequence $\left\{\frac{a}{b} - \frac{1}{n}\right\}$ shows that $f\left(\frac{a}{b}^-\right)$ sts. Therefore, f is discontinuous and left and right limit exists f has a simple discontinuity at every rational point.

) Suppose f is a real function with domain $\mathbb{R}$ which has the ermediate value property: If $f(a) < c < f(b)$ then $f(x) = c$ for some between a and b. Suppose also, for every rational r, that the set of all with $f(x) = r$ is closed. ($S = \{x : f(x) = r\}$ is closed)

Prove that f is continuous.

Following the hint: The proof is by contradiction.
Suppose f is not continuous. then, there exist a sequence $\{x_n\}$ s.t. $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$, so by the density of the rational numbers on $\mathbb{R}$ we can conclude that $f(x_n) > r > f(x_0)$ for some r and all n, then consider the sequence $\{t_n\}$, we will have that $f(t_n) = r$ for some $x_0 < t_n < x_n$. Hence, by squeezing theorem we have that $t_n \to x_0$.
But $\{t_n\}$ is closed by hypothesis since $f(t_n) = r$. Hence, $\{t_n\}$ should contain all of its limits points. In particular $\{t_n\}$ should contain x_0, but that would mean that $f(x_0) = r$, a contradiction with $f(x_0) < r$. therefore, f has to be continuous.

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(21) Suppose K and F are disjoint sets in a metric space X, where K is compact and F is closed. Prove that there exists $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$ and $q \in F$. Show that the conclusion fa[ils] for two disjoint closed sets if neither is compact

Pf: By contradiction: Suppose that

$$\forall \delta > 0 : \exists x \in K \text{ and } y \in F : \; d(x,y) < \delta.$$

Let $\delta_n = \frac{1}{n}$. Define $\{x_n\}$ and $\{y_n\}$ by picking points for each $\delta_1, \delta_2, \ldots$
Now, K is compact. By theorem proved in class we know that K is sequentially compact. Since $\{x_n\} \subset K$, we can conclude that there exist $\{x_{n_k}\} \subset K$ and $x_0 \in K$ s.t. $x_{n_k} \to x_0$.
But then $d(x_n, y_n) < \delta = \frac{1}{n}$. Look at $\{y_{n_k}\}$, the corresponding subsequence of $\{y_n\}$ that matches the subsequence $\{x_{n_k}\}$. By hypothesis $d(x_{n_k}, y_{n_k}) < \frac{1}{n}$. In particular this, together with $x_{n_k} \to x_0$ imply

$$d(x_0, y_{n_k}) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y_{n_k}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for choices of $\delta = \frac{1}{n} < \frac{\varepsilon}{2}$ (for large enough n). But then $y_{n_k} \to x_0$, Hence x_0 is a limit point of F (because $\{y_{n_k}\} \subset F$)
Since F is closed we must have $x_0 \in F$. We also have $x_0 \in K$. therefore

$x_0 \in K \cap F$, but $K \cap F = \phi$, a contradiction. therefore,

$\exists \delta > 0 : \forall x \in K, y \in F : \quad d(x,y) > \delta$ (the result we wanted!)

ow to show that the conclusion fails if the two disjoint, closed sets are neither compact.

onsider:

$$A = \{ n + \tfrac{1}{n} : n \in \mathbb{N} \}. \qquad B = \mathbb{N}.$$

ow, both of these sets are closed since neither one has any mit points. Moreover, $A \cap B = \phi$, for if $x \in A \cap B$ then

$x = n + \frac{1}{n} = n$, which is a contradiction.

inally, neither A nor B are compact. To see this, choose the pen cover of sets to be themselves.

ow, the conclusion fails because

$$\left| \left(n + \tfrac{1}{n}\right) - n \right| = \frac{1}{n} \to 0,$$ So there exists no such δ as proved before.