M413 - Analysis I - Fall 2013 - Enrique Areyan - HW2 30/30 ①

ⓐ Prove that given a real number x, either $x>0$ or $x=0$ or $x<0$.

Remark: let us prove that $x<0$ if and only if $-x>0$.
By Homework 1. If $x<0$ then $x-x<0-x$, which is the same as $-x>0$
Conversely, If $-x>0$ then $x-x>x$, which is the same as $x<0$.
Hence, we can reformulate our initial statement as follow:
Prove that given a real number x, either $x>0$ or $x=0$ or $-x>0$.
Pf: Let x be a real number and let $\{x_n\}\in \mathcal{C}_{\mathbb{Q}}$ be a representative of x in $\mathcal{C}_{\mathbb{Q}}/\sim$, i.e., $\{x_n\}\in[x]$. the proof consists of two parts:
(I) Prove that at least one of the statements: $x>0$, $x=0$, $-x>0$ is true
(II) Prove that no two of the previous statements can be true at the same time

(I) Suppose that both $x>0$ and $-x>0$ are false.
We want to show that $x\overset{?}{=}0$, which is the same as $x_n\to 0$ as $n\to\infty$
(taking the sequence of all zeros $\{0\}\in[0]$, then $x_n-0\to 0 \iff x_n\to 0$).
By definition $\{x_n\}$ is Cauchy. Hence, given $\varepsilon>0:\exists N:\forall n,m\geq N: |x_n-x_m|<\frac{\varepsilon}{2}$
By assumption it is not the case that $x>0$. Hence: $\forall\varepsilon>0\ \forall N:\exists n\geq N: x_n<\varepsilon$
Likewise, it is not the case that $-x>0$. Hence: $\forall\varepsilon>0:\forall N:\exists n\geq N: -x_n<\varepsilon$

Let $\varepsilon>0$. Pick N and $n,m\geq N$ such that

$$|x_n-x_m|<\frac{\varepsilon}{2} \iff -\frac{\varepsilon}{2}<x_n-x_m<\frac{\varepsilon}{2}$$

$\iff x_n<x_m+\frac{\varepsilon}{2}$ and $x_n>x_m-\frac{\varepsilon}{2}$. But, we can bound x_m:

Let $\varepsilon'=\frac{\varepsilon}{2}>0$. then, for any N (in particular the same as before), there exists $m\geq N$ such that: $x_m<\varepsilon'$ and $-x_m<\varepsilon'$. Use this m and N, with m...
together with the above equations to get:

$$x_n<x_m+\frac{\varepsilon}{2}<\varepsilon'+\frac{\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \implies \boxed{x_n<\varepsilon},\text{ and:}$$

$$x_n>x_m-\frac{\varepsilon}{2}>-\varepsilon'-\frac{\varepsilon}{2}=-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=-\varepsilon \implies \boxed{x_n>-\varepsilon}$$

(since $-x_m<\varepsilon' \iff x_m>-\varepsilon'$)

therefore, $-\varepsilon < x_n < \varepsilon \iff |x_n| < \varepsilon$, so this shows that $x_n \to 0$ as $n \to \infty$,
and so $\{x_n\} \in [x]$, which means $x = 0$. this concludes part (I) of the proof.
Hence, at least one of the statements $x > 0$ or $x = 0$ or $-x > 0$ is true.

(II) To prove that two of the statement cannot both be true simultaneously
we need to consider 3 cases:

(1) Suppose $x = 0$ and $x > 0$. then $\{x_n\} \in [0]$, and in particular we can
pick the sequence $\{0\} \in [0]$ as a representative for x. By definition of $x > 0$:
$\exists \varepsilon > 0 : \exists N : \forall n \geq N : x_n \geq \varepsilon$. Now, let $\varepsilon > 0$ and pick N and $n \geq N$. s.t.

$x_n \geq \varepsilon > 0 \Rightarrow x_n > 0$; but our representative for x_n is $x_n = 0$
for any n. So $x_n = 0 > 0 \Rightarrow 0 > 0$ a contradiction in the order of the
rational numbers. therefore, the two statements $x = 0$ and $x > 0$ cannot
both be true.

(2) Suppose $x = 0$ and $-x > 0$. claim: $x = 0$ if and only if $-x = 0$ Pf of claim
If $x = 0$. then $\{x_n\} \in [0]$, i.e., $x_n \to 0$ as $n \to 0$. Now, let $\varepsilon > 0$.
pick N and $n \geq N$ s.t: $|x_n| < \varepsilon$. But then $|-x_n| = |x_n| < \varepsilon \Rightarrow |-x_n| < \varepsilon$
. So $-x_n \to 0$ as $n \to \infty$, with $\{-x_n\} \in [-x]$. therefore, $-x = 0$. the other direction
follows trivially from this proof. Hence, $x = 0 \iff -x = 0$. (End of claim).
Now, apply (1) to $y = -x$ to get that the two statements
$y = -x = 0 = x$ and $y = -x > 0$ cannot both be true.

(3) Suppose $x > 0$ and $-x > 0$. Let $\{x_n\} \in [x]$, $\{-x_n\} \in [x]$. By definition:
$\exists \varepsilon > 0 : \exists N : \forall n \geq N : x_n \geq \varepsilon$
$\exists \varepsilon' > 0 : \exists N' : \forall n \geq N' : -x_n \geq \varepsilon'$
Choose $\varepsilon > 0$ and $\varepsilon' > 0$ and N, N' and $n \geq \max(N, N')$ such that.
then

$x_n \geq \varepsilon > 0$ and $-x_n \geq \varepsilon' > 0$.

$x_n > 0$ and $-x_n > 0$. But adding this equations over rational numbers:

$x_n - x_n > 0 + 0 \iff 0 > 0$. a contradiction in the order
of rational numbers. therefore, the two statements $x > 0$ and $-x > 0$
cannot both be true.

Parts (I) and (II) of the proof show that for a given real number x, <u>either</u> $x > 0$ OR $x = 0$ OR $-x > 0$ ($\Leftrightarrow x < 0$) □

ⓑ Prove that given real numbers x, y, either $x > y$ or $x = y$ or $x < y$

<u>Pf</u>: Let $z = y - x$. We have proved that addition of real numbers is closed. We also know of existence of additive inverse of real numbers. Hence, $z \in \mathbb{R}$. By part ⓐ we can conclude that:

<u>either</u> $z > 0$ OR $z = 0$ OR $-z > 0$.

But, by definition each of these statements are equivalent to:

$$z > 0 \Leftrightarrow y - x > 0 \Leftrightarrow y > x$$
$$z = 0 \Leftrightarrow y - x = 0 \Leftrightarrow y = x$$
$$-z > 0 \Leftrightarrow -(y - x) > 0 \Leftrightarrow x - y > 0 \Leftrightarrow x > y.$$

this proves part ⓑ.

ⓒ Prove that every Cauchy sequence of real numbers converges to a real number.

<u>Pf</u>: Let $\{x_n\}$ be a cauchy sequence of real numbers. By definition of being Cauchy we have that given $\varepsilon > 0 : \exists N : \forall n, m \geq N : |x_n - x_m| < \varepsilon$

Now, let $\varepsilon > 0$. Pick N and $n, m \geq N$ such that $|x_n - x_m| < \frac{\varepsilon}{2}$

Define $S = \{t \in \mathbb{R} \mid t < x_n\}$. this is ① non-empty and ② bounded above s

① Note that we prove a fact about cauchy sequences of rational numbers, namely: every Cauchy sequence of rational numbers is bounded. this is true for cauchy sequences of real numbers and the proof of this is very similar to that of cauchy sequences of rational numbers. I ommit the proof here and use this fact.

Since $\{x_n\}$ is bounded, $\exists M$ s.t. $|x_n| < M \;\; \forall n$. this means that $-M < x_n$ and $x_n < M$. So $-M \in S \Rightarrow S \neq \emptyset$ and ② S is bounded

above since $M > x_n$, for all n.
① and ② allow us to use the Completeness theorem and conclude that S has a least upper bound. Let $\ell = \sup S$.
We want to prove that $x_n \overset{?}{\to} \ell$ as $n \to \infty$.
By previous argument, $|x_n - x_m| < \frac{\varepsilon}{2}$. In particular, set $m = N+1$. then

$|x_n - x_{N+1}| < \frac{\varepsilon}{2} \iff -\frac{\varepsilon}{2} < x_n - x_{N+1} < \frac{\varepsilon}{2}$, and so we get:

$$\begin{cases} x_n < x_{N+1} + \frac{\varepsilon}{2} \iff x_{N+1} + \frac{\varepsilon}{2} \notin S \iff \ell \leq x_{N+1} + \frac{\varepsilon}{2} \iff \ell - x_{N+1} \leq \frac{\varepsilon}{2} \\ x_n > x_{N+1} - \frac{\varepsilon}{2} \iff x_{N+1} - \frac{\varepsilon}{2} \in S \iff \ell \geq x_{N+1} - \frac{\varepsilon}{2} \iff \ell - x_{N+1} \geq -\frac{\varepsilon}{2} \end{cases}$$

$\Rightarrow -\frac{\varepsilon}{2} \leq \ell - x_{N+1} \leq \frac{\varepsilon}{2} \iff |\ell - x_{N+1}| \leq \frac{\varepsilon}{2} \iff |x_{n+1} - \ell| \leq \frac{\varepsilon}{2}$

Now we can use this bound and the fact that $\{x_n\}$ is Cauchy as follow:

$$\begin{aligned} |x_n - \ell| &= |x_n - \ell + x_{N+1} - x_{N+1}| \\ &= |x_n - x_{N+1} + x_{N+1} - \ell| \\ &\leq |x_n - x_{N+1}| + |x_{N+1} - \ell| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

$\Rightarrow$ Given $\varepsilon > 0$, there exists N such that

$|x_n - \ell| \leq \varepsilon$, provided that $n \geq N$.

This shows that for an arbitrary cauchy sequence of real numbers, the sequence converges to a real number ℓ. □