M413 - Enrique Areyan - Fall 2013 - HW4 (1)

All exercises from Rudin, page 43. 50/50

(5) Construct a bounded set of real numbers with exactly three limit points

Solution: Let $S=\{\frac{1}{n}\}\cup\{1+\frac{1}{n}\}\cup\{2+\frac{1}{n}\}$, for $n=1,2,3,\ldots$.

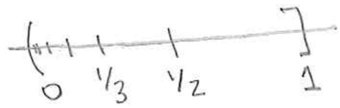
claim: the set S has exactly three limit point, namely $0, 1$ and 2. Moreov[er] this set is bounded.

Pf: that S is bounded is clear. For instance, take $r=5$ and consider $N_r(0)$ then, $N_r(0)=(-5,5)$ and $S\subset N_r(0)$. So S is bounded.

To prove that S has exactly three limit points, we can prove that each pie[ce] of S has exactly one limit point and we'll be done. Namely, let us show that (i) $\{\frac{1}{n}\}$ has exactly one limit point (0), (ii) $\{1+\frac{1}{n}\}$ has exactly one limi[t] point (1) and (iii) $\{2+\frac{1}{n}\}$ has exactly one limit point (2).

(i) First, 0 is a limit point of $\{\frac{1}{n}\}$. this follows from the fact that $\{\frac{1}{n}\}\to 0$ as $n\to\infty$. By definition of convergence, $\forall\varepsilon>0: \exists N: |\frac{1}{n}|<\varepsilon, \forall n\geq N$.

So, choose $\varepsilon>0$ and N so that the previous hold. then, let $r=\varepsilon$, $y=\frac{1}{N}<r \Rightarrow y\in N_r(0)$ and $y\neq 0$. therefore $N_r(0)\setminus\{0\}\cap\{\frac{1}{n}\}\neq\Phi$ for any $r>0$. Hence, 0 is a limit point of $\{\frac{1}{n}\}$.



Next, there are no other limit points on $\{\frac{1}{n}\}$ over $\mathbb{R}$. Since the rationals are dense in the reals, we can pick two real numbers tha[t] are between any rational number. Let r_1, r_2 be such that $\frac{1}{n+1}<r_1<\frac{1}{n}$ and $\frac{1}{n}<r_2<\frac{1}{n-1}$, for $n=2,3,\ldots$ (1 is obviously not a limit point: take $r=\frac{1}{4}$, then $N_{1/4}(1)\setminus\{1\}\cap\{\frac{1}{n}\}=\Phi$). Now, pick $r=\min(r_1,r_2)$. then, $r>0$ and fix m, we have $N_r(\frac{1}{m})\setminus\{\frac{1}{m}\}\cap\{\frac{1}{n}\}=\Phi$; so that any of the points $\frac{1}{m}$, $m=1,2,\ldots$ is not a limit [point].

Likewise, for any real number between two consecutive terms of $\frac{1}{n}$, we can choose $r=\frac{1}{2}\min(d(x,\frac{1}{n}), d(x,\frac{1}{n+1}))$ So that $N_r(x)\setminus\{x\}\cap\{\frac{1}{n}\}=\Phi$, so that any real number between 0 and 1 is not a limit point of $\{\frac{1}{n}\}$. 10

therefore, the set $\{\frac{1}{n}\}$, for $n=1,2,\ldots$ has exactly one limit point, namely 0.

(ii) and (iii) follow in a very similar fashion using almost identical argument[s] since $\{1+\frac{1}{n}\}, \{2+\frac{1}{n}\}$ are just shifts of $\{\frac{1}{n}\}$ to the right by 1 and 2 respectivel[y].

herefore, $S=\{\frac{1}{n}\}\cup\{1+\frac{1}{n}\}\cup\{2+\frac{1}{n}\}$, $n=1,2,\dots$ is a bounded set of real numbers
ith exactly three limit points.

) Let E' be the set of all limit points of a set E.

(a) Prove that E' is closed.

(b) Prove that E and $\bar{E}$ have the same limit points

(c) Do E and E' always have the same limit points?

olution: For (a) I can think of two strategies:

(1) Prove that $(E')^c$ is open thereby showing that $((E')^c)^c = E'$ is closed.

(2) Show that every limit point of E' is a point of E', which in this cases means that every limit point of E' is a limit point of E.

 am going to use strategy (2).

t p be a limit point of E'. then, by definition, $\forall r>0: N_r(p)\setminus\{p\}\cap E'\neq\emptyset$.
t $r>0$. Pick q a point such that $q\in N_r(p)\setminus\{p\}\cap E'$. then, $q\in N_r(p)$ and
$\neq p$ and $q\in E'$. By this last fact we know that q is a limit point
 E. So, for any $r_1>0$: $N_{r_1}(q)\setminus\{q\}\cap E\neq\emptyset$. Choose $r_1=\frac{1}{2}(r-d(p,q))$. then,
$_{r_1}(q)\subset N_r(p)$. Pick $s\in N_{r_1}(q)\setminus\{q\}\cap E$. then $s\in N_{r_1}(q)\Rightarrow s\in N_r(p)$ and
$\neq q$ and $s\neq p$ and $s\in E$. therefore, $N_r(p)\setminus\{p\}\cap E\neq\emptyset$ (in particular
$\in N_r(p)\setminus\{p\}\cap E$), which means that p is a limit point of E, which
 definition means that $p\in E'$. □

 (b), Let us show that $E'=(\bar{E})'$, as usual by double containment.

) Let $x\in E'$ then x is a limit point of E. therefore, for any $r>0$,
$(x)\setminus\{x\}\cap E\neq\emptyset$. Let $y\in N_r(x)\setminus\{x\}\cap E$. then $y\in N_r(x)\setminus\{x\}$ and $y\in E$.
e $y\in E\Rightarrow y\in\bar{E}$. therefore, $N_r(x)\setminus\{x\}\cap\bar{E}\neq\emptyset$. So, x is a limit point
 $\bar{E}$, which means that $x\in(\bar{E})'$.

 Let $x\in(\bar{E})'$. then x is a limit point of $\bar{E}$. therefore, for any $r>0$,
$x)\setminus\{x\}\cap\bar{E}\neq\emptyset \Leftrightarrow N_r(x)\setminus\{x\}\cap(E\cup E')\neq\emptyset$. let $q\in N_r(x)\setminus\{x\}\cap(E\cup E')$.
en, $q\in N_r(x)\setminus\{x\}\cap E$ or $q\in N_r(x)\setminus\{x\}\cap E'$. In the first case we have
nd that x is a limit point of E, therefore $x\in E'$ and we are done.
 the second case, we have that x is a limit point of E'. But in (a) we
wed that every limit point of E' is a limit point of E. Hence, $x\in E'$. □

For ⓒ consider the following counter example:
Define $E=\{\frac{1}{n}\}$ for $n=1,2,\dots$ We proved in exercise (5) that
E has only 0 as it limit points, i.e., $E'=\{0\}$. However,
E' has no limit points for suppose E' has a limit point p.
then, for any $r>0$, $N_r(p)\setminus\{p\}\cap E'\neq\emptyset$. Let $x\in N_r(p)\setminus\{p\}\cap E'$
$\Rightarrow x\in N_r(p)\setminus\{p\}$ and $x\in E'$. this final statement means $x\in\{0\}\Rightarrow x=$
therefore $0\in N_r(p)\setminus\{p\}$. Clearly, 0 cannot be a limit point. Moreov
we can always choose r small enough so that $0\notin N_r(p)\setminus\{p\}$, by
picking $r<d(0,p)$. therefore, there is no such limit point p and so
$(E')'=\emptyset\neq E'=\{0\}$, so E and E' do not have to have the same lim. poi

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(7) Let $A_1, A_2, A_3,\dots$ be subsets of a metric space. [First I am going to check that this definition make sens]
(a) If $B_n=\bigcup_{i=1}^{n} A_i$, prove that $\overline{B_n}=\bigcup_{i=1}^{n}\overline{A_i}$, for $n=1,2,3,\dots$

Pf: (⊆) Let $x\in\overline{B_n}$. then $x\in B_n\cup B_n'$. $\Rightarrow x\in B_n$ or $x\in B_n'$

If $x\in B_n$ then $x\in\bigcup_{i=1}^{n}A_i\Rightarrow x\in A_k$, for some k, $1\le k\le n$. Hence, $x\in A_k\cup A_k'$
and so $x\in\overline{A_k}$. therefore, $\boxed{x\in\bigcup_{i=1}^{n}\overline{A_i}}$.

otherwise, If $x\in B_n'$ then x is a limit point of B_n, so x is a limit
point of $\bigcup_{i=1}^{n}A_i$ so for every $r>0$: $N_r(x)\setminus\{x\}\cap(\bigcup_{i=1}^{n}A_i)\neq\emptyset$; therefore,
there exists k, $1\le k\le n$, such that $N_r(x)\setminus\{x\}\cap A_k\neq\emptyset$, otherwise the
union will be empty which is not the case. But then x is a limit point
of A_k $\Rightarrow x\in A_k'\Rightarrow x\in A_k\cup A_k'\Rightarrow x\in\overline{A_k}\Rightarrow\boxed{x\in\bigcup_{i=1}^{n}\overline{A_i}}$

(⊇) Let $x\in\bigcup_{i=1}^{n}\overline{A_i}$. then $x\in\overline{A_k}$, for some k, $1\le k\le n$. Hence, $x\in A_k\cup A_k'$

If $x\in A_k$ then $x\in\bigcup_{i=1}^{n}A_i\Rightarrow x\in B_n\Rightarrow x\in B_n\cup B_n'\Rightarrow\boxed{x\in\overline{B_n}}$

If $x\in A_k'$ then x is a limit point of A_k, for some $1\le k\le n$, by an argument similar to the case (⊆), x is a limit point of $\bigcup_{i=1}^{n}A_i$, so x is a limit point of $B_n\Rightarrow x\in B_n'\Rightarrow x\in B_n\cup B_n'\Rightarrow\boxed{x\in\overline{B_n}}$

So, these definitions make sense in the finite union case. Now, let us prove ⓐ for all $n\in\mathbb{N}$ by induction:

Induction: Consider the statement $S(n)$: If $B_n = \bigcup_{i=1}^{n} A_i$ then $\overline{B_n} = \bigcup_{i=1}^{n} \overline{A_i}$.

BASE CASE: $S(n=1)$: If $B_1 = \bigcup_{i=1}^{1} A_1$ then $\overline{B_1} = \bigcup_{i=1}^{1} \overline{A_i}$.

Suppose $B_1 = \bigcup_{i=1}^{1} A_1 = A_1$. then $\overline{B_1} = \overline{A_1} = \bigcup_{i=1}^{1} \overline{A_i}$. So base case holds.

Inductive STEP: Suppose $S(n)$ is true. We want to prove $S(n+1)$, i.e.,

$S(n+1)$: If $B_{n+1} = \bigcup_{i=1}^{n+1} A_i$ then $\overline{B_{n+1}} = \bigcup_{i=1}^{n+1} \overline{A_i}$

: Suppose $B_{n+1} = \bigcup_{i=1}^{n+1} A_i$ and $B_n = \bigcup_{i=1}^{n} A_i$

$\overline{B_{n+1}} = \overline{\left(\bigcup_{i=1}^{n+1} A_i\right)}$ by assumption

$= \overline{\left[\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right]}$ Separating the union.

$= \overline{B_n \cup A_{n+1}}$ by assumption

$= \overline{B_n} \cup \overline{A_{n+1}}$ by inductive hypothesis, considering $C = B_n \cup A_{n+1}$,

$= \bigcup_{i=1}^{n} \overline{A_i} \cup \overline{A_{n+1}}$ by inductive hypothesis

$= \bigcup_{i=1}^{n} \overline{A_i}$ by combining the union.

If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$

: Let $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$. then, $x \in \overline{A_K}$, for some K.

Hence, $x \in A_K \cup A_K'$, so $x \in A_K$ or $x \in A_K'$

$x \in A_K$ then $x \in \bigcup_{i=1}^{\infty} A_i \Rightarrow x \in B \Rightarrow x \in B \cup B' \Rightarrow x \in \overline{B}$

$x \in A_K'$ then x is a limit point of A_K. So, for any $r > 0$.

$(x) \setminus \{x\} \cap A_K \neq \emptyset$, so $N_r(x) \setminus \{x\} \cap \left(\bigcup_{i=1}^{\infty} A_i\right) \neq \emptyset$, so x is a limit

nt of $\bigcup_{i=1}^{\infty} A_i = B$ (by hypothesis). therefore, $x \in B' \Rightarrow x \in B \cup B'$

$x \in \overline{B}$.

the result holds, i.e., $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$ □

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Show, by an example that this inclusion can be proper.

Using ideas developed in (5) and (6), consider $B = \{\frac{1}{n}, n \in \mathbb{N}\}$.

then, B can be decomposed as $B = \bigcup_{i=1}^{\infty} A_i$, where $A_i = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{i}\}$

If you fix an i, then A_i contains finitely many points, so A_i is closed

thus, $\overline{A_i} = A_i$. But, we already proved in (5) that $B' = \{0\}$, so $\overline{B} = B \cup \{0$

therefore $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$, But $\{0\} \not\subset \overline{A_i}$ for any i, so $\{0\} \not\subset \bigcup_{i=1}^{\infty} \overline{A_i}$, so the

inclusion can be proper.

(8) Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E?

Solution: YES. Pf: Let $E \subset \mathbb{R}^2$ be an open set. By definition, all points

in E are interior to E. Let $x \in E$ then, there exist $r > 0$ such that

$N_r(x) \subset E$. Now, let $r_1 > 0$. Pick a point $y \in \mathbb{R}^2$ such that

this choice is possible because the space is $\mathbb{R}^2$.

$$d(x,y) < \frac{1}{2}\min(r, r_1).$$

Now, $d(x,y) > 0 \Rightarrow x \neq y$. Moreover, by our choice of $d(x,y)$ we have that

$y \in N_{r_1}(x)$ and $y \in N_r(x) \subset E \Rightarrow y \in E$. therefore,

$N_{r_1}(x) \setminus \{x\} \cap E \neq \emptyset$ (in particular q is in this intersection

Hence, x is a limit point of E. $\square$

Answer the same question for closed sets in $\mathbb{R}^2$.

Solution: NO. Consider the set $E = \{(1,2)\}$. cleary this set is closed

since it contains all of its limit points, i.e., $E' = \emptyset$. Moreover, $(1,2) \in \mathbb{R}$

is not a limit point of E.

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