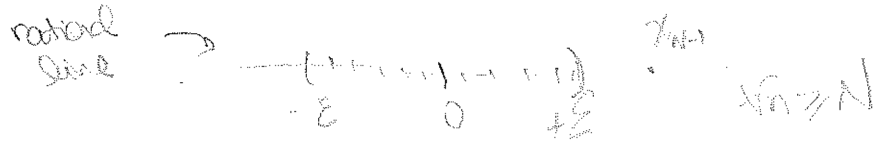
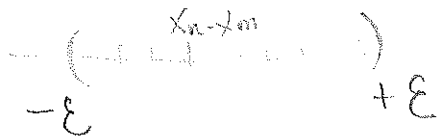
Analysis I Enrique Areyan - Fall 2013

Def: We say that $x_n \to 0$ as $n \to \infty$ if given $\varepsilon > 0$, $\exists N$ (rational, integer) s.t.
$$|x_n| < \varepsilon \quad \text{provided that } n \geq N.$$



$x_n \to l$ as $n \to \infty$, l rational, if $x_n - l \to 0$ as $n \to \infty$; i.e., given $\varepsilon > 0$ $\exists N$ s.t. $|x_n - l| < \varepsilon$ $\forall n \geq N$.

Def: We say that a sequence x is Cauchy if given $\varepsilon > 0$ (rational) $\exists N$ (integer) s.t.
$$|x_n - x_m| < \varepsilon \quad \forall n, m \geq N.$$



Property: ⓐ Convergent sequences are cauchy.
ⓑ Not all cauchy sequences are convergent (in $\mathbb{Q}$).

Pf: ⓐ Let x be a convergent sequence s.t. $x_n \to l$. By definition given $\varepsilon > 0$ $\exists N$: $|x_n - l| < \varepsilon$ whenever $n \geq N$. In particular pick N s.t. $|x_n - l| < \frac{\varepsilon}{2} < \varepsilon$.
We want to show that x is cauchy. Suppose $n, m \geq N$

$$|x_n - x_m| = |x_n - x_m - l + l| = |x_n - l + l - x_m| \leq |x_n - l| + |l - x_m| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow |x_n - x_m| \leq \varepsilon$, so x is cauchy.

ⓑ Consider the sequence $1, 1.4, 1.41, 1.414, 1.4142, \dots$
this is a cauchy sequence but it does not converge in $\mathbb{Q}$.

ⓒ Cauchy sequences are bound. $\exists M$ (rational) s.t. $|x_n| \leq M$ $\forall n$

Pf: Let x be a cauchy sequence. Fix $\varepsilon > 0$. Find N s.t. $|x_n - x_m| < \varepsilon = a$ $\forall n, m \geq N$
$\varepsilon = a$ here $|x_n| \leq \dots$

$|x_n| \leq M = \max(|x_1|, |x_2|, \dots, |x_{N-1}|, |x_N|, |x_N| + a)$ $\forall n$

$$|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| \leq |x_N| + a$$

list of numbers $x_1, x_2, \dots x_N + a$

Defining real numbers:

Let $\mathcal{C}_{\mathbb{Q}} = \{\text{all rational Cauchy sequences}\}$. Let us define the relation $\sim$ on $\mathcal{C}_{\mathbb{Q}}$ as follow: for $x, y \in \mathcal{C}_{\mathbb{Q}}$: $x \sim y$ iff $x_n - y_n \to 0$ as $n \to \infty$.

Claim: $\sim$ is an equivalence relation (RST).

Pf: (i) Reflexivity: Let $x \in \mathcal{C}_{\mathbb{Q}}$. $x_n - x_n = 0$ for any n. Therefore, $x_n - x_n \to 0$ as $n \to \infty$ (in fact it is always zero). This means that $x \sim x$.

(ii) Symmetry: Let $x, y \in \mathcal{C}_{\mathbb{Q}}$. Suppose $x \sim y$, i.e., $x_n - y_n \to 0$ as $n \to \infty$. By def. given $\varepsilon > 0$ $\exists N$ s.t. $|x_n - y_n| < \varepsilon$ whenever $n \geq N$. By properties of absolute value $|x_n - y_n| = |y_n - x_n| < \varepsilon$ whenever $n \geq N$. Hence, $y_n - x_n \to 0$ as $n \to \infty$ which means that $y \sim x$.

(iii) Transitivity: Let $x, y, z \in \mathcal{C}_{\mathbb{Q}}$. Suppose $x \sim y$ and $y \sim z$. By definition of $\sim$:

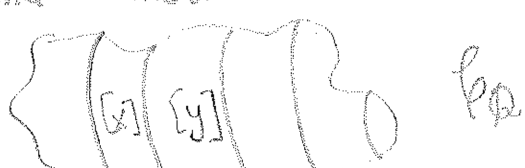
Given $\varepsilon > 0$: $\begin{cases} \exists N_1 \text{ s.t. } |x_n - y_n| < \frac{\varepsilon}{2} < \varepsilon \text{ whenever } n \geq N_1 \\ \exists N_2 \text{ s.t. } |y_n - z_n| < \frac{\varepsilon}{2} < \varepsilon \text{ whenever } n \geq N_2 \end{cases}$

Pick n such that the following two conditions above hold, i.e., $n \geq \max(N_1, N_2)$

$$|x_n - z_n| = |x_n - y_n + y_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$$
$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow |x_n - z_n| < \varepsilon$, $n \geq \max(N_1, N_2)$. Therefore $x_n - z_n \to 0$, so $x \sim z$. The relation $\sim$ is an equivalence relation.

$[x] \cap [y] = \phi$ or $[x] = [y]$



Def: the real numbers are the equivalence classes of $\mathcal{C}_{\mathbb{Q}}/\sim$

Properties of real numbers: $+, \cdot, ">"$

$+$: Given $[x], [y]$, we want to show that the sum is an element of $\mathcal{C}_{\mathbb{Q}}/\sim$, $[x+y]$

Let $x, x' \in [x]$ and $y, y' \in [y]$. Is the following sequence Cauchy? $(x_n + y_n), (x_n' + y_n')$

Given $\varepsilon > 0$ pick N_1 s.t. $|x_n - x_n'| \leq \frac{\varepsilon}{2} < \varepsilon$ and pick N_2 s.t. $|y_n - y_n'| \leq \frac{\varepsilon}{2} < \varepsilon$

Let $n \geq \max(N_1, N_2)$. Then: $|(x_n + y_n) - (x_n' + y_n')| = |(x_n - x_n') + (y_n - y_n')|$
$\leq |x_n - x_n'| + |y_n - y_n'|$

Hence, the sum is closed. $x + y \in [x+y]$

Properties of real numbers:

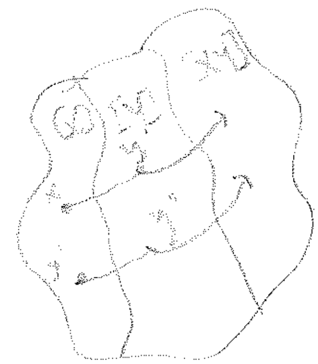
$\cdot$: Given $[X], [Y] \Rightarrow [XY]$. (Construct a class:)

$X \cdot Y$ should also be an equivalence class.

By hypothesis: $x_n - x_n' \to 0$ as $n \to \infty$, $y_n - y_n' \to 0$ as $n \to \infty$

Let $\{x_n\}, \{x_n'\} \in [X]$ and $\{y_n\}, \{y_n'\} \in [Y]$.

$$\begin{aligned}
|x_n \cdot y_n - x_n' \cdot y_n'| &= |x_n \cdot y_n - x_n y_n' + x_n y_n' - x_n' \cdot y_n'| \\
&= |x_n(y_n - y_n') + y_n'(x_n - x_n')| \\
&\leq |x_n(y_n - y_n')| + |y_n'(x_n - x_n')| \\
&= |x_n||y_n - y_n'| + |y_n'||x_n - x_n'| \\
&\leq |x_n| \varepsilon_1 + |y_n'| \varepsilon_2 \\
&\leq M_1 \varepsilon_1 + M_2 \varepsilon_2 = \varepsilon
\end{aligned}$$



Choose $\varepsilon \geq M_1\varepsilon_1 + M_2\varepsilon_2$

Property: $[X] \neq [0]$ real, then we can find y s.t. $[X][Y] = [1]$

In order to prove this property, first identify Cauchy sequences that are not zero.

If $[X] \neq [0]$, $\{x_n\}$ cauchy, then $\exists N$ s.t. $|x_n| \geq \varepsilon > 0$ $\forall n \geq N$.

Pf: ⓐ Consider the subsequence: $\eta > 0$, $\{n_k\}$ s.t. $|x_{n_k}| \geq \eta$, $\forall n_k$

ⓑ By def $\{x_n\}$ is cauchy, i.e., given $\delta > 0$ $\exists N$ s.t.: $|x_n - x_m| \leq \delta$ $\forall n, m \geq N$

Let $\delta = \frac{\eta}{2}$. Find N for that value. Next, pick $n_k > N$

$$\begin{aligned}
|x_n| = |x_n - x_{n_k} + x_{n_k}| = |x_n - x_{n_k} - (-x_{n_k})| &\geq ||x_n - x_{n_k}| - |-x_{n_k}|| \\
&= ||x_{n_k}| - |x_n - x_{n_k}|| \\
&\geq \eta - \eta/2 = \eta/2 = \varepsilon \quad \square
\end{aligned}$$

Proof of $[X][Y] = [1]$:

Pick $[Y]$ s.t. $y_1 = \cdots = y_N = 0$; $y_n = \frac{1}{x_n}$, $n > N$. $\Rightarrow [XY]$ $\overbrace{0, \ldots, 0}^{1 \ldots N}, \overbrace{1, 1, \ldots}^{\geq N}$ $[1]$

Need to prove that y is a Cauchy sequence, i.e., $y_n - y_m = \frac{1}{x_n} - \frac{1}{x_m} = \frac{x_m - x_n}{x_n x_m}$

ⓐ Pick N s.t. $|x_n| \geq \eta$, $\forall n \geq N$ (since $[X] \neq 0$)

ⓑ Given $\delta > 0$, specifically, $\delta = \eta^2 \varepsilon$, $\exists N'$ s.t. $|x_n - x_m| \leq \delta$, $\forall n, m \geq N'$

Let $M = \max(N, N')$ and $n, m \geq M$. Then: $|y_n - y_m| = \frac{|x_n - x_m|}{|x_n||x_m|} \leq \frac{1}{\eta^2} |x_n - x_m| \leq \frac{1}{\eta^2} \eta^2 \varepsilon = \varepsilon$ $\square$

Properties of $\mathbb{R}$:

1. Order
2. Every Cauchy sequence of real numbers converge. (in constrast with $\ell(\mathbb{Q})$). } Completeness
3. Suppose $S \neq \emptyset$, set of real numbers bounded above. then S has a sup = l.u.b. (least upper bound)
($\mathbb{Q}$ does not have this property, think of $1.4, 1.41, 1.414, \ldots \to \sqrt{2} \notin \mathbb{Q}$). } Completeness

Order $x > 0$ if: $x \neq 0$ with $\{x_n\}$ is s.t $x_n > 0$ $\forall n \geq N$. Observe that in this case $\exists$ a natural number $\varepsilon > 0$ s.t. $x_n \geq \varepsilon$ $\forall n \geq N$. (why?).

Define $x > y$ if $x - y > 0$, i.e., $x - y \neq 0$, $\{x_n - y\}$ is s.t. $x_n - y_n > 0$ $\forall n \geq N$.
Now we can define:
$$|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$
(look at 0 as the sequence $0, 0, 0, \ldots 0, \ldots$
$0 > x \Leftrightarrow 0 - x > 0 \Leftrightarrow -x > 0$

Archimedian Property: let x, y be reals. $x, y > 0$. then, $\exists$ a positive integer m s.t. $m x > y$

Pf: let $x, y \in \mathbb{R}$, $x, y > 0$. We want to find a positive integer m such that: $mx > y \Leftrightarrow mx - y > 0 \Leftrightarrow \exists N, m$: $mx_n - y_n > 0$ $\forall n \geq N$. Note that here we are taking m to be the constant sequence $\{m\}$.

Proof by contradiction: Suppose that $\forall N, m$: $\exists n \geq N$: $mx_n - y_n \leq 0 \Leftrightarrow mx_n \leq y_n$
Let K (positive integer) be a bound for y_n. Since x_n, y_n are cauchy sequences, they are bounded.
then: $mx_n \leq y_n \leq K \Leftrightarrow mx_n \leq K$ (*).
Look at (*) for $m = 1$. $\exists n_1$: $x_{n_1} \leq K$.
Look at (*) for $m = 2$, $N = n_1$. then $\exists n_2$: $n_2 > n_1$ s.t. $2x_{n_2} \leq K$.
Look at (*) for $m = 3$, $N = n_2$. then $\exists n_3$. $n_3 > n_2 > n_1$ s.t $3 x_{n_3} \leq K$.
⋮ (by induction)
Look at (*) for $m = \ell$, $N = n_{\ell-1}$. then $\exists n_\ell$: $n_\ell > n_{\ell-1} > \cdots > n_1$ s.t. $\ell x_{n_\ell} \leq K$.
Pick such n_ℓ. then $\ell x_{n_\ell} \leq K$. Since K, ℓ are integers, $\frac{K}{\ell} \in \mathbb{Q}$. So
$$x_{n_\ell} \leq \frac{K}{\ell}, \text{ for any } \ell.$$

therefore,
① By assumption $x > 0$. Hence $\{x_n\}$ is Cauchy s.t. $\exists \varepsilon > 0$ s.t. $x_n \geq \varepsilon$ $\forall n \geq N$.
② $\{x_{n_\ell}\}_{n_\ell \to \infty}$ is s.t. $x_{n_\ell} \leq \frac{K}{\ell} < \varepsilon$ whenever $\ell > \frac{K}{\varepsilon}$.

contradiction: $x_{n_\ell} < \varepsilon$, $x_n \geq \varepsilon$ to the following

□

This is a cauchy sequence with a convergent subsequence but it does not converge.

Every Cauchy sequence with a convergent subsequence must converge to the same number.

Pf: Let $\{x_n\}$ be a Cauchy sequence with a subsequence $\{x_{n_k}\} \to \ell$ as $n_k \to \infty$.
Want to prove that $\{x_n\} \to \ell$, i.e., $\forall \overset{\text{rational}}{\varepsilon > 0}\ \exists \overset{\text{integer}}{N}: |x_n - \ell| < \varepsilon$, when $n \geq N$.
We know that given $\varepsilon > 0\ \exists N'$ s.t. $|x_{n_k} - \ell| < \varepsilon$ when $n_k \geq N'$.

Now, given $\varepsilon > 0$: $\begin{cases} \text{Pick } N \text{ s.t } |x_n - x_{n_k}| < \frac{\varepsilon}{2}, \forall n, n_k \geq N & (\text{since } x_n \text{ is Cauchy}) \\ \text{pick } N' \text{ s.t } |x_{n_k} - \ell| < \frac{\varepsilon}{2}, \forall n_k \geq N' & (\text{since } \{x_{n_k}\} \to \ell) \end{cases}$

Pick $n, n_k \geq M = \max(N, N')$. then:

$$\begin{aligned} |x_n - \ell| &= |x_n - \ell + x_{n_k} - x_{n_k}| \\ &= |x_n - x_{n_k} + x_{n_k} - \ell| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - \ell| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence, Given $\varepsilon > 0$ $\exists M = \max(N, N')$ s.t. $|x_n - \ell| < \varepsilon$, so that $\{x_n\} \to \ell$.

Property: Rationals are dense in $\mathbb{R}$.
Given $x \in \mathbb{R}$ and $\overset{\text{rational or real}}{\varepsilon > 0}$: $\exists r \in \mathbb{Q}: |x - r| < \varepsilon$. Pf: Pick $\overset{\text{a representative of } x.}{\{x_n\}} \in [x]$. then we have:
Given $\overset{\text{rational}}{\varepsilon > 0}$: $\exists \overset{\text{integer}}{N}$ s.t: $|x_n - x_m| \leq \varepsilon\ \forall n, m \geq N$. (since $\{x_n\}$ is Cauchy).
Let $\varepsilon > 0$. Pick N and $n, m \geq N$ s.t: $|x_n - x_m| \leq \varepsilon$. Now, let $m = N$. then,
Define r a rational number to be $r = x_N, x_N, \ldots = \{x_N\}$.

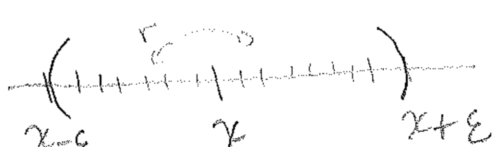
If $x - r > 0$, then $|x - r| = x - r = x_1 - x_N, x_2 - x_N, \ldots$
$= x_n - x_N < \varepsilon$

If $x - r < 0$, then $|x - r| = r - x = x_N - x_1, x_2 - x_N, \ldots$
$= x_N - x_n < \varepsilon$

If $x - r = 0 \Rightarrow x = r$, then $|x - r| = 0 < \varepsilon$.

In any case $|x - r| < \varepsilon$. So we have found $r = \{x_N\}$.

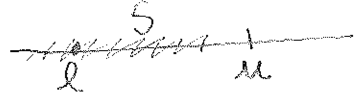
In picture:



<u>Completeness</u>: Let S be a non-empty set of real numbers bounded above. (i.e., $\exists M^{\text{irrational}}$ s.t $\forall s \in S: s < M$). then S has a <u>least upper bound</u> $= \sup S$.

<u>Pf</u>: (First note that this is not true for $S \subset \mathbb{Q}$, e.g. $S = \{r \in \mathbb{Q} \mid r^2 < 2\}$. this set is such that $\sup S = \sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$).

<u>Summary of proof</u>:



① Pick u_1 to be an upper bound for S and l_1 not an upper bound for S.

② Show that u_1, l_1 can be picked as integers. Use archimedian principle.

③ Define inductively (recursively) sequences $\{u_n\}, \{l_n\}$ such that, u_n is always and upper bound and l_n is always not an upper bound. Show that $u_1 \geq u_2 \geq \cdots \geq u_n \geq l_n \geq \cdots \geq l_1$ holds using induction.

④ Show that $\{u_n\}$ and $\{l_n\}$ are cauchy sequences.

⑤ Show that $u_n - l_n \to 0$ then this are equivalent cauchy sequences.

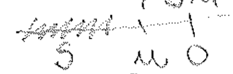
⑥ Consider B to be the real number that both $\{u_n\}$ and $\{l_n\}$ converge to. Show that: (i) B is an upper bound; (ii) B is the least upper bound ($\sup S$).

<u>Details of proof</u>:

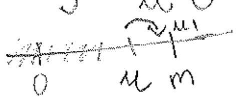
① Since S is bounded above we can always pick u_1 to be an upper bound and l_1 to not be an upper bound.

② u_1, l_1 can be picked integers. Two cases:

For u $\begin{cases} \text{If } u \leq 0, \\ \text{o/w } u > 0 \end{cases}$



, choose $u_1 = 0$ an integer. $u_1 = 0 \geq u$, so u_1 is upper bound.



, by archimedian principle, take $1, u$; then $\exists m$ (integer) s.t $1 \cdot m = m > u$. Choose $u_1 = m$, still an upper bound.

In either case u can be picked an integer u_1.

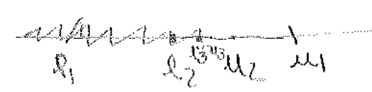
For l $\begin{cases} \text{If } l \geq 0 \\ \text{o/w } l < 0 \end{cases}$

, choose $l_1 = 0$ an integer. $l_1 = 0 \leq l$, so l_1 is not an upper bound.

$-l > 0$. by archimedian principle, take $1, -l$, then $\exists m$ (integer) s.t. $1 \cdot m = m > -l$. then $-m < l$. Choose $l_1 = -m$, still not an upper bound.

In either case l can be picked an integer l_1. these are picked so that $u_1 \geq l_1$.

③ Define inductively sequences $\{u_n\}, \{l_n\}$, By the rule:



If $\frac{u_n + l_n}{2}$ is an upper bound for S, <u>then</u> $l_{n+1} = l_n$ and $u_{n+1} = \frac{u_n + l_n}{2}$

o/w If $\frac{u_n + l_n}{2}$ is NOT an upper bound for S, <u>then</u> $l_{n+1} = \frac{u_n + l_n}{2}$ and $u_{n+1} = u_n$

Note that by construction u_n remains an upper bound for S for any n while l_n remains not an upper bound for S for any n.

Now, we need to prove by induction that: $u_n \geq u_{n+1}$, $l_n \leq l_{n+1}$ and $u_{n+1} \geq l_{n+1}$. In this manner we can have the total order $u_1 \geq u_2 \geq \cdots \geq u_n \geq l_n \geq l_{n-1} \geq \cdots \geq l_1$

<u>BASE CASE</u>: $n=1$. Want to prove: $u_1 \overset{?}{\geq} u_2$, $l_1 \overset{?}{\leq} l_2$, $u_2 \overset{?}{\geq} l_2$.

<u>If</u> $\frac{u_1+l_1}{2}$ is an upper bound for S. <u>then</u> $l_2 = l_1$ and $u_2 = \frac{u_1+l_1}{2}$. In this case:

$\boxed{l_1 \leq l_2}$ (trivially) and $u_2 = \frac{u_1+l_1}{2} = \frac{u_1}{2} + \frac{l_1}{2} \leq \frac{u_1}{2} + \frac{u_1}{2} = u_1 \Rightarrow \boxed{u_2 \leq u_1}$ (By our choice $u_1 \geq l_1$)

and $u_2 = \frac{u_1+l_1}{2} = \frac{u_1}{2} + \frac{l_1}{2} \geq \frac{l_1}{2} + \frac{l_1}{2} = l_1 \Rightarrow u_2 \geq l_1 = l_2 \Rightarrow \boxed{u_2 \geq l_2}$

<u>O/w</u> $\frac{u_1+l_1}{2}$ is not an upper bound for S. <u>then</u> $l_2 = \frac{u_1+l_1}{2}$ and $u_2 = u_1$. In this case:

$\boxed{u_1 \geq u_2}$ (trivially) and $l_2 = \frac{u_1+l_1}{2} = \frac{u_1}{2} + \frac{l_1}{2} \geq \frac{l_1}{2} + \frac{l_1}{2} = l_1 \Rightarrow \boxed{l_1 \leq l_2}$ (By our choice $u_1 \geq l_1$)

and $l_2 = \frac{u_1+l_1}{2} = \frac{u_1}{2} + \frac{l_1}{2} \leq \frac{u_1}{2} + \frac{u_1}{2} = u_1 \Rightarrow u_1 = u_2 \geq l_2 \Rightarrow \boxed{u_2 \geq l_2}$ this shows the base case.

<u>Inductive STEP</u>: Suppose that the result hold up to n for $n \geq 1$. then: $u_n \geq u_{n-1}$, $l_{n-1} \leq l_n$ and $u_n \geq l_n$. We want to show the result for $n+1$: $u_{n+1} \overset{?}{\geq} u_n$, $l_n \overset{?}{\leq} l_{n+1}$ and $u_{n+1} \overset{?}{\geq} l_{n+1}$.

<u>If</u> $\frac{u_n+l_n}{2}$ is an upper bound for S, <u>then</u>: $l_{n+1} = l_n$ and $u_{n+1} = \frac{u_n+l_n}{2}$. In this case:

$\boxed{l_n \leq l_{n+1}}$ (trivially) and $u_{n+1} = \frac{u_n+l_n}{2} = \frac{u_n}{2} + \frac{l_n}{2} \leq \frac{u_n}{2} + \frac{u_n}{2} = u_n \Rightarrow \boxed{u_{n+1} \leq u_n}$ (By hyp $u_n \geq l_n$).

and $u_{n+1} = \frac{u_n+l_n}{2} = \frac{u_n}{2} + \frac{l_n}{2} \geq \frac{l_n}{2} + \frac{l_n}{2} = l_n = l_{n+1} \Rightarrow \boxed{u_{n+1} \geq l_{n+1}}$.

<u>O/w</u> $\frac{u_n+l_n}{2}$ is not an upper bound for S. <u>then</u> $l_{n+1} = \frac{u_n+l_n}{2}$ and $u_{n+1} = u_n$. In this case:

$\boxed{u_n \geq u_{n+1}}$ (trivially) and $l_{n+1} = \frac{u_n+l_n}{2} = \frac{u_n}{2} + \frac{l_n}{2} \geq \frac{l_n}{2} + \frac{l_n}{2} = l_n \Rightarrow \boxed{l_n \leq l_{n+1}}$ (By hyp. $u_n \geq l_n$)

and $l_{n+1} = \frac{u_n+l_n}{2} = \frac{u_n}{2} + \frac{l_n}{2} \leq \frac{u_n}{2} + \frac{u_n}{2} = u_n = u_{n+1} \Rightarrow \boxed{u_{n+1} \geq l_{n+1}}$. this shows the result for $n \in \mathbb{N}$.

(4) We want to show that $\{u_n\}, \{l_n\}$ are cauchy sequences.

<u>First</u>, write $u_n - l_n$ in closed form:

$$u_n - l_n = \begin{cases} u_{n-1} - \frac{u_{n-1}+l_{n-1}}{2} = \frac{u_{n-1}-l_{n-1}}{2} \\ \frac{u_{n-1}+l_{n-1}}{2} - l_{n-1} = \frac{u_{n-1}-l_{n-1}}{2} \end{cases} \left\{\begin{matrix} \frac{u_{n-2} - \frac{u_{n-2}+l_{n-2}}{2}}{2} = \frac{u_{n-2}-l_{n-2}}{2^2} \\ \frac{\frac{u_{n-2}+l_{n-2}}{2} - l_{n-2}}{2} = \frac{u_{n-2}-l_{n-2}}{2^2} \end{matrix}\right\} = \cdots = \frac{u_1-l_1}{2^{n-1}}$$

this should be proved by induction.

$$\Rightarrow u_n - l_n = \frac{u_1-l_1}{2^{n-1}}.$$

Let $\varepsilon > 0$. Pick N and $n, m \geq N$ s.t. $\frac{u_1-l_1}{2^{N-1}} \leq \varepsilon$. then.
↳ note that $u_1 \geq l_1 \Rightarrow \frac{u_1-l_1}{2^{N-1}} \geq 0$

$$l_n \underset{\text{By construction}}{\leq} u_m \Rightarrow -u_m \leq -l_n \Rightarrow u_n - u_m \leq u_n - l_n \underset{\text{Part (b) proved}}{=} \frac{u_1-l_1}{2^{N-1}} \leq \varepsilon.$$

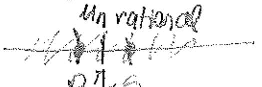
therefore, $u_n - u_m \leq \varepsilon$

Likewise, $l_m \leq u_n \Rightarrow l_m - l_n \leq u_n - l_n = \frac{u_1-l_1}{2^{N-1}} \leq \varepsilon \Rightarrow l_m - l_n \leq \varepsilon$.

(5) ... equivalent cauchy sequences.

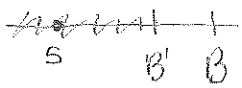
⑥ Since $u_n - l_n \to 0$ and $\{u_n\}, \{l_n\}$ are Cauchy, they are representatives of the same real number. Call this number B. then $\{u_n\}, \{l_n\} \in [B]$. For this number we want to show:

(i) B is and upper bound. Suppose not. then, let $s \in S$ be such that $B < S$



Use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. then $\exists r \in \mathbb{Q}$ s.t. $B < r < S$.
But, $r - B > 0$; Now, think of the sequences attached to this numbers.
$r \to r, r, r, \ldots$, $B \to u_1, u_2, u_3, \ldots, u_n, \ldots$, $r - B \to r - u_1, r - u_2, \ldots$. then, $\exists N : \forall n \geq N \;\; r - u_n > 0$
$\Rightarrow r > u_n$ But then $S > r > u_n \Rightarrow S > u_n$, a contradiction since u_n is always an upper bound.

(ii) B is the least upper bound. Suppose not. then there exists another smaller bound B'



. then, $B - B' > 0$. Like before, think of the sequences.
$\exists N : \forall n \geq N : l_n - B' > 0 \Rightarrow l_n > B' \overset{\text{since } B' \text{ is upper bound}}{>} S \Rightarrow l_n > S$ a contradiction since l_n is not an upper bound.

Hence, ①, ②, ③, ④, ⑤, ⑥ $\to$ Completeness.

Note that an equivalent formulation is that of bounded below, $S \neq \emptyset$, then S has a greatest lower bound $=$ inf S. To prove this map $S \to -S$ and apply the other version of completeness we have just proved.

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theorem 1.21: let $n \geq 2, x > 0$. then, $\exists y \in \mathbb{R}$ s.t. $y^n = x$.

Pf: Define $S = \{t \in \mathbb{R} \mid t^n < x\}$. ⓐ S is not empty since $0^n = 0 < x$, $0 \in S$.
ⓑ S is bounded above. two cases:

(i) $x \leq 1$. Choose any number $u > 1$, then $u^n \overset{\text{Power is an increasing function since } u > 1}{>} 1 \geq x$; $u \notin S$, u an upper bound.

(ii) $x > 1$. choose any number $u > x$ since $u^n > u > x$; $u \notin S$, u an upper bound.

Combining (i) and (ii), we can choose the bound $u = 1 + x$, so $u \notin S$ an upper bound.
By completeness, S has a least upper bound. Let $y = \sup S$.
We proved that given $a, b \in \mathbb{R}$ then either $a > b$ or $b > a$ or $a = b$.
To complete our proof, we want to show that $x = y^n$. to do this, let us show that the two cases $x > y^n$ and $y^n > x$ lead to a contradiction.

Before: we need the fact that If $b > a > 0$, and n is an integer:

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots + ba^{n-2} + a^{n-1}) \quad (b > a)$$
$$< (b-a)(b^{n-1} + b^{n-2}b + b^{n-3}b^2 + \cdots + bb^{n-2} + b^{n-1})$$
$$= (b-a)(b^{n-1} + b^{n-1} + b^{n-1} + \cdots + b^{n-1} + b^{n-1})$$
$$= (b-a)\, n\, b^{n-1}$$

$$\Rightarrow \boxed{b^n - a^n < (b-a)\, n\, b^{n-1}} \quad (\star)$$

Now, let us show that $y^n > x$, $x > y^n$ separetaly each leads to a contra.

since $0 \in S$ and $y = \sup S$.

(I) If $x > y^n$. Choose $0 < h < 1$. Note that $y+h > y > 0$

Apply (*) with $b = y+h$ and $a = y$:

since $h < 1$.

$$(y+h)^n - y^n < (y+h-y)n(y+h)^{n-1} = hn(y+h)^{n-1} < hn(y+1)^{n-1}$$

We can make $hn(y+1)^{n-1}$ as small as we want provided h is small.

By archimedian property: $\exists N$ s.t. $1, n(y+1)^{n-1}$. $N = N \cdot 1 > n(y+1)^{n-1}$

Multiply by h: $hn(y+1)^{n-1} < hN$. Pick h rational to be $h = \frac{1}{N\ell}$

$x - y^n > 0 > \frac{1}{\ell}$ for some $\ell \in \mathbb{Z}$

then $hn(y+1)^{n-1} < \frac{N}{N\ell} = \frac{1}{\ell}$

$$(y+h)^n - y^n < hn(y+1)^{n-1} < \frac{1}{\ell} \overset{?}{=} x - y^n \quad \text{for an appropiate } h.$$

$\Rightarrow (y+h)^n - y^n < x - y^n \Rightarrow (y+h)^n < x$, hence, $y+h \in S$. But $y+h > y$ but y is the least upper bound so $y > y+h$ a contradiction.

$y > K > 0$. since $y^n > x \Rightarrow y^n - x > 0 \Rightarrow \frac{y^n - x}{ny^{n-1}} = K > 0$

(II) If $y^n > x$. Choose $K = \frac{y^n - x}{ny^{n-1}}$.

$$y^n - (y-K)^n < (y - y + K)\, n y^{n-1} = K n y^{n-1} = \frac{y^n - x}{ny^{n-1}} \cdot ny^{n-1} = y^n - x$$

$\Rightarrow y^n - (y-K)^n < y^n - x \Rightarrow (y-K)^n > x$ Hence, $y - K \notin S$. therefore $y-K$ is an upper bound but $y - K < y$ and y is the least upper bound. Contradiction.

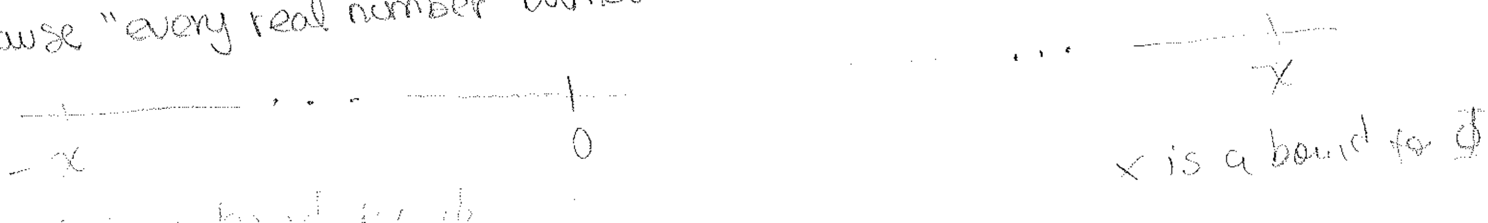
this shows that $y^n = x$. (Note that a similar argument works for any increasing function)

<u>Extended real number system</u>: add to the reals two symbols: $+\infty$ and $-\infty$.

these are going to be forced upon bounds: $\forall x \in \mathbb{R}$: $-\infty < x < +\infty$

these have arithmetic rules (page 12 Rudin). (Not that $\infty + (-\infty)$ is not defined) BEWARE!

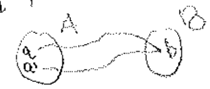
As a remark, let $S = \emptyset$. then $\sup S = -\infty$ and $\inf S = +\infty$, this is because "every real number bounds the empty set"



Analysis I. Enrique Areyan - Fall 2013.

CHAPTER 2: BASIC topology

Let A, B be sets. $f: A \to B$ is a function.
$E \subset A$, $f(E) = \{f(x) \in B \mid \text{for } x \in E\}$, the image of E under f.
$f(A)$ is the range, clearly $f(A) \subseteq B$. If $f(A) = B$, f is onto.
$E \subset B$, $f^{-1}(E) = \{a \in A \mid f(a) \in E\}$. Note that for an element $b \in B$, $f^{-1}(b)$ may not be well-defined, However this is always well-defined, $f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}$.



$f(\{b\}) = \{a, a'\}$. f is 1-1 if $f^{-1}(\{b\}) = \{a\}$. Alternatives definition for 1-1 and onto are:
1-1: $\forall x, y \in X: f(x) = f(y) \Rightarrow x = y$; onto: $\forall y \in Y: \exists x \in X: f(x) = y$.

Definition: Let $\sim$ over the sets of all sets be defined as follow: $A \sim B$ iff there exist $f: A \to B$ which is 1-1 and onto. this is an equivalence relation, i.e., reflexive: $A \sim A$ (use identity function). Symmetry: If $A \sim B$ then $B \sim A$ (use the inverse f^{-1} of the function f provided for $A \sim B$). transitivity: If $A \sim B$ and $B \sim C$ then $A \sim C$ (use composition of functions). ($A \sim B$ iff have same cardinal number; same number of elem.)

Definition: $J_n = \{1, 2, \dots, n\}$. J = set of all positive integers.
(a) A is finite if $A \sim J_n$ for some n. (b) A is infinite if A is not finite (c) A is countable if $A \sim J$. (d) A is uncountable if A is neither finite nor countable.

Remark: there is no largest cardinal number.

Example: $\mathbb{N} \sim \{\text{even integers}\}$ since $f: \mathbb{N} \to \{\text{even integers}\}$ given by $f(n) = 2n$ is clearly 1-1 and onto.
this example shows that the arithmetic of cardinal numbers behaves different than since $\aleph_0$ = cardinality of natural numbers = cardinality of even integers and
$\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.

Also, this example shows that while a finite set cannot be equivalent with one of its proper subsets, this may be true for infinite sets.

Definition: a sequence is a function f defined on the set J of all positive integers $f(n) = x_n$, for $n \in J$, we denote the sequence f by $\{x_n\}$ or $x_1, x_2, x_3, \dots$
Note that the terms $x_1, x_2, x_3, \dots$ may not be distinct.
... any countable set can be arranged in a sequence.

theorem: If A is an infinite, countable set and $E \subset A$ is infinite the E is countable.

Pf: Since A is countable, its elements can be listed $\{a_n\}$, i.e., $a_1, a_2, a_3, \dots, a_n, \dots$
Let $N_1 = \{n \in J : a_n \in E\} \neq \Phi$ By the least integer axiom, pick n_1 to be the least positive integer in N_1 (the first element in N_1). Now, let $N_2 = \{n \in J : a_n \in J$ and $n > n_1\}$. Again, by least integer axiom, pick n_2 to be the first element in N_2.
Continue this process. let $N_k = \{n \in J : a_n \in E$ and $n > n_{k-1}\}$, where n_{k-1} is the least integer in N_{k-1}. then, let $f: J \to E$ defined as $f(k) = a_{n_k}$, for $k = 1, 2, 3, \dots$, so f is a 1-1 and onto map between J and E.

$$\begin{matrix} a_{n_1}, & a_{n_2}, & \dots, & a_{n_k}, & \dots \\ \uparrow & \uparrow & & \uparrow & \\ 1 & 2 & \dots & k & \dots \end{matrix}$$

Remark: in a sense, countable sets represent the "smallest" infinity. No uncountable set can be a subset of a countable set.

Rationals are countable:
First work with positive rationals, $\{\frac{m}{n}\}, m, n \in \mathbb{Z}^+, n \neq 0$. Define $f: \mathbb{Q}^+ \to J$ by $f(\frac{m}{n}) = 2^m 3^n$ (any two coprime numbers will work)
the function is 1-1: let $\frac{m_1}{n_1}, \frac{m_2}{n_2}$ be such that $f(\frac{m_1}{n_1}) = f(\frac{m_2}{n_2}) \Rightarrow 2^{m_1}3^{n_1} = 2^{m_2}3^{n_2}$; since $\gcd(2,3) = 1$, we get that dividing by 2^{m_1} $3^{n_1} = 3^{n_2} \Rightarrow n_1 = n_2$, likewise, dividing by 3^{n_1} we get $2^{m_1} = 2^{m_2} \Rightarrow m_1 = m_2$. therefore $\frac{m_1}{n_1} = \frac{m_2}{n_2}$.
the function is onto a subset of J. Clearly, the range of f is infinite, since we can make m, n as large as we want. Call Range of f: $f(\mathbb{Q}^+) = J^*$ then $J^* \subset J$. Note that this is a proper subset since a number like 11 will never be divisible by 2 or 3 and hence $11 \notin J^*$.
But, $f: \mathbb{Q}^+ \to J^*$ is a bijection. Hence $\mathbb{Q}^+ \sim J^*$.
Moreover, $J^* \subset J$ is an infinite subset and by definition J is countable. By previous theorem J^* is countable, i.e., $J^* \sim J$. By transitivity of $\sim$
$\mathbb{Q}^+ \sim J^*$ and $J^* \sim J \Rightarrow \mathbb{Q}^+ \sim J$ and thus, $\mathbb{Q}^+$ is countable. $\mathbb{Q} = \mathbb{Q}^- \cup \mathbb{Q}^+$
A similar argument works for $\mathbb{Q}^-$. And then, by next theorem is countable.

theorem: let $\{A_m\}, m = 1, 2, 3, \dots$ be a collection (family) of countable sets.
then, $\bigcup_{m=1}^{\infty} A_m = \{a : a \in A_m$ for some $m\}$ is also countable.

Pf: Arrange the sets A_m and its elements as follow (which we can do since A_j is countable)

$A_1: a_{11}\ a_{12}\ a_{13} \dots$
$A_2: a_{21}\ a_{22}\ a_{23} \dots$
$A_3: a_{31}\ a_{32}\ a_{33} \dots$
$\vdots$
$A_n: a_{n1}\ a_{n2}\ a_{n3} \dots$

this array contains all elements of S.
Arrange elements of S in the sequence
$a_{11}; a_{21}, a_{12}; a_3, a_{22}\, a_{13}; \dots$

there might be repetitions, i.e., an element that is inside E_i and E_j for $i \neq j$. So, these will appear more than once in the sequence. Since S is the union, these repetitions are not in S. But, there is a subset $T \subset J$, such that $S \sim T$ (just ommit repetitions in the sequence and use the sequence as your bijection "$f(n) = a_n$"). But now, $T \subset J$, T infinite and by definition J is countable so $T \sim J$ (T is countable). So we have $S \sim T$ and $T \sim J$, by transitivity, $S \sim J$ and so S is countable. Note that this has no contradiction: $E_i \subset S$ and S countable $\Rightarrow E_i$ countable.

<u>Proposition</u>: there exists a set which is <u>not</u> countable.

<u>Pf</u>: Let $S = \{$all sequences with terms $0, 1\}$. the S is not countable. Suppose for a contradiction that S is countable. then we can form the sequence of elements of S: $\{S_n\}$, $S_1, S_2, S_3, \ldots$ Arrange this as:

$S_1: 0\ 1\ 0\ 0\ 1\ 0 \cdots$
$S_2: 1\ 0\ 1\ 0\ 0\ 0 \cdots$
$S_3: 0\ 0\ 1\ 0\ 0\ 0 \cdots$
$\vdots$
$S_n: 1\ 1\ 1 \cdots 0\ 0\ 1 \cdots$
$\vdots$

Look at the diagonal (Cantor's diagonal).

Construct $S_d = 1\ 1\ 0 \cdots 1 \cdots$, i.e., switching ones and zeros for elements in the diagonal. Formally:

$$S_{d_i} = \begin{cases} 1 & \text{if } S_{i_i} = 0 \\ 0 & \text{otherwise} \end{cases} \quad ; i = 1, 2, 3, \ldots$$

But then $S_d \notin S_j$ for any j. But S_d is a sequence of 0's and 1's So we have constructed a sequence that is not in S, but this is a contradiction with the definition of S. therefore, S is uncountable.

<u>Note</u>: this construction implies that real numbers are uncountable thinking of real numbers with base 2 instead of 10. the only technicality is that of sequences $1000\cdots0 = 01111\cdots$.

<u>Metric Spaces</u>: once $\mathbb{R}$ is constructed, a simple question is: How far apart are x and y? $\overset{}{\underset{x}{+}}\ \underset{y}{+}$, for distance we do not care if x is to the left or right of y. A natural function to use for measuring distance in $\mathbb{R}$ is the absolute value $|\,|: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$. this satisfies:

$|x-y| = 0$ iff $x = y$. otherwise $|x-y| > 0$. Also, $|x-y| = |y-x|$.

$|x-z| \leq |x-y| + |y-z|$

In general, let X be any set. Let $d: X \times X \to \mathbb{R}^+$. We say that (X, d) is a <u>metric space</u> provided that d satisfies:

① $d(p,p) = 0 \quad \forall p \in X$, $d(p,q) > 0$, $p \neq q \in X$.

② $d(p,q) = d(q,p)$

③ $d(p,q) \leq d(p,r) + d(r,q)$. Triangular inequality.

<u>Def</u>: A <u>Ball</u> in a metric space centered at $p \in X$, with radius $r > 0$ is

$\{q \in X : d(p,q) < r\}$ Open ball

$\{q \in X : d(p,q) \leq r\}$ closed ball

<u>Def</u>: A set $E \subset X$ of a metric space X is convex if, $\forall p, q \in E$ and $0 \leq \lambda \leq 1$, $\lambda \in \mathbb{R}$.

$$\lambda p + (1-\lambda) q \in E$$

<u>Example</u>. open balls are convex. <u>Pf</u>: Consider the open ball B of radius r, centered at $x \in X$. Want to show B is convex. Let $y, z \in B$. By definition $|y-x| < r$ and $|z-x| < r$. But then: Let $0 \leq \lambda \leq 1$

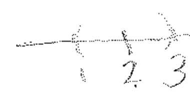
$$\begin{aligned}
|\lambda y + (1-\lambda) z - x| &= |\lambda(y-x) + (1-\lambda)(z-x)| \\
&\leq |\lambda(y-x)| + |(1-\lambda)(z-x)| \quad \text{(triangular Ineq.)} \\
&= \lambda|y-x| + (1-\lambda)|z-x| \quad (\text{since } 0 \leq \lambda \leq 1) \\
&< \lambda r + (1-\lambda) r \quad \text{(By hypothesis)} \\
&= \lambda r + r - \lambda r \\
&= r
\end{aligned}$$

$\Rightarrow \lambda y + (1-\lambda) z \in B$

$\Rightarrow B$ is convex.

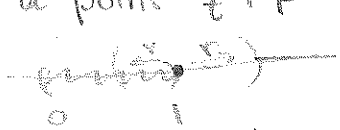
<u>Definitions</u>. Let (X, d) be a metric space,

(a) A <u>neighborhood</u> of p is $N_r(p) = \{q \in X \mid d(p,q) < r\}$, for some $r > 0$. r is called the radius of $N_r(p)$. <u>Ex</u>: $N_1(2) = \{q \in \mathbb{R} \mid |2-q| < 1\} = (1,3)$



(b) A point p is a <u>limit point</u> of $E \subset X$ if every $N_r(p)$ contains a point $q \neq p$ such that $q \in E$.

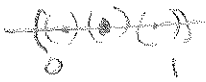
<u>Ex</u>:



1 is a limit point of $(0,1)$.

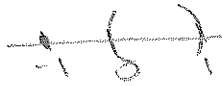
However, any number > 1 is not a limit point b/c you can find $r > 0$ s.t. $N_r(p)$ does not contain any element of $(0,1)$.

Also note that any point inside $(0,1)$ is a limit point



limit points can be "approximated" by points in E.

(c) If $p \in E$ and p is not a limit point of E the p is an <u>isolated point</u> of E.
In other words, $p \in E$ is isolated if $\exists r > 0$ s.t. $E \cap N_r(p) = \{p\}$.

<u>Ex</u>: $S = \{-1\} \cup (0,1)$.  ; -1 is an isolated point. e.g., $N_1(-1) \cap (0,1) = \{-1\}$

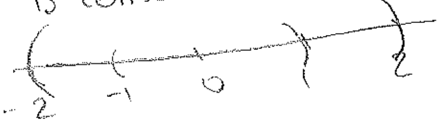
(d) E is <u>closed</u> if every limit point of E belongs to E
<u>Ex</u>: Any finite set will be closed. In particular a point is close because it has no limit points.

(e) A point p is an <u>interior point</u> of E if $\exists r > 0$ s.t. $N_r(p) \subset E$
<u>Ex</u>: the interior of $[0,1]$ is $(0,1)$

(f) E is <u>open</u> if all of its points are interior.
<u>Ex</u>: $(0,1)$.

(g) $E^c = \{q \in X \mid q \notin E\}$, <u>complement</u> of E

(h) E is <u>perfect</u> if E is closed and all of its points are limit points of E
A perfect set contains no isolated points. <u>Ex</u>: $[0,1]$, <u>non example</u>: $[0,1] \cup \{2\}$.

(i) E is <u>bounded</u> if $\exists p, R$, s.t. $E \subset N_R(p)$, i.e., if E is contained in a ball of finite radius. <u>Ex</u>: $E = (-1,1)$ is bonded. take $R = 2$, $p = 0$ 

(j) E is <u>dense</u> in X if every point of X belongs to E OR is a limit point of E. <u>Ex</u>. rational numbers are dense in $\mathbb{R}$.

<u>2.19</u>: Every neighborhood is an open set.

<u>Pf</u>: $N_r(p)$: 

Want to show that any point $q \in N_r(p)$ is an interior point of $N_r(p)$
An interior point q of $N_r(p)$ is s.t. there exists $r_1 > 0$ s.t. $N_{r_1}(q) \subset E$.
We have that $d(p,q) = s < r$

Pick $r_1 = r - s$. Let $x \in N_{r_1}(q) \iff d(x,q) < r_1 = r-s$. W.t.S. $x \overset{?}{\in} N_r(p) \iff d(x,p) \overset{?}{<} r$

$d(x,p) \leq d(x,q) + d(q,p)$ Triangular inequality.
$< r - s + s = r$ By $d(q,q) = s$ & $d(x,q) < r-s$.

$\Rightarrow d(x,p) < r \iff x \in N_r(p) \iff x$ is an interior point of $N_r(p) \iff N_r(p)$ is an open set.

<u>2.20</u>: <u>If</u> p is a limit point of E, <u>then</u> every neighborhood of p contains infinitely many points of E

<u>Pf</u>: By contradiction. Let p be a limit point of E. Suppose there exists a neighborhood of p that contains finitely many points of E. List these points $q_1, q_2, \ldots, q_n$ then, there exists $r > 0$ s.t. $d(p,q_1) < r$, $d(p,q_2) < r$, $\ldots$, $d(p,q_n) < r$. Now, take the minimum $h = \min(d(p,q_1), d(p,q_2), \ldots, d(p,q_n))$. But then,

[illegible] E. a contradiction with p being a limit point of E.

2.22: Let $\{E_\alpha\}$ be a collection of sets E_α. then: $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$

Pf: $x \in \left(\bigcup_\alpha E_\alpha\right)^c \Leftrightarrow x \notin \bigcup_\alpha E_\alpha \Leftrightarrow x \notin E_\alpha \overset{\forall\alpha}{\Leftrightarrow} x \in E_\alpha^c \overset{\forall\alpha}{\Leftrightarrow} x \in \bigcap_\alpha E_\alpha^c$

Note: α could be an uncountable index

2.23: theorem: E is open iff E^c is closed.

Pf: ($\Leftarrow$) Want to show that every point $x \in E$ is interior to E.
Let $x \in E \Rightarrow x \notin E^c$. since E^c is closed, x is not a limit point of E^c.
therefore, there exists a neighborhood $N_r(x)$ such that $E^c \cap N_r(x) = \Phi$, therefore $N_r(x) \subset E$, which means that x is an interior point

($\Rightarrow$) Want to show that every point $x \in E^c$ is a limit point of E^c.
Let x be a limit point of E^c. then, Every neiborhood of x contains a point $q \in E^c$
Hence, x is not an interior point of E. Since E is open we conclude that $x \in E^c$
So E^c contains all of its limit points. therefore E^c is closed. ◻
Note that an equivalent formulation of this theorem is E is closed iff E^c is open.

Properties: Let X be a metric space. Let $\{G_\alpha\}$ be a collection of open sets (G = German for open).
Let $\{F_\beta\}$ be a collection of closed sets (F = French for closed). (α, β might be uncountable).

ⓐ $\bigcup_\alpha G_\alpha$ is open

ⓑ $\bigcap_\alpha F_\beta$ is closed

ⓒ $\bigcap_{k=1}^m G_k$ is open

ⓓ $\bigcup_{k=1}^m F_\beta$ is closed

So, Union of open is open. Intersection of closed is closed.
BEWARE OF THE opposite.
Intersection of open might not be open: EX $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ $(n = 1,2,3)$. G_n is open.
But $\bigcap_{n=1}^\infty G_n = \{0\}$ which is not open (it is actually closed).
Union of closed might not be closed. EX $F_n = \left[1 - \frac{1}{n}, 2\right]$. $(n = 1,2,3)$. F_n is closed.
But $\bigcup_{n=1}^\infty F_n = (1,2]$, which is not closed.

Pf: ⓐ we want to show that $\bigcup_\alpha G_\alpha$ is open, i.e., $x \in \bigcup_\alpha G_\alpha$ is an interior point
Let $x \in \bigcup_\alpha G_\alpha \Rightarrow x \in G_\alpha$ for some $\alpha \Rightarrow x$ is interior to $G_\alpha \Rightarrow \exists N_r(x) \subset G_\alpha$
$\Rightarrow N_r(x) \in \bigcup_\alpha G_\alpha \Rightarrow x$ is interior to $\bigcup_\alpha G_\alpha \Rightarrow \bigcup_\alpha G_\alpha$ is open.

Now, ⓐ $\Rightarrow$ ⓑ Since if $\bigcup_\alpha G_\alpha$ is open, then $\left(\bigcup_\alpha G_\alpha\right)^c$ is closed. But,
By previous theorem $\left(\bigcup_\alpha G_\alpha\right)^c = \bigcap_\alpha G_\alpha^c$; and since G_α is open $G_\alpha^c = F_\alpha$ is closed.
therefore $\bigcap F_\alpha$ is closed

Pf of ⓒ: We want to show that $\bigcap_{k=1}^{m} G_k$ is open., i.e., $x \in \bigcap_{k=1}^{m} G_k$ is interior to $\bigcap_{k=1}^{m} G_k$

Let $x \in \bigcap_{k=1}^{m} G_k$. then $x \in G_k$ for any k. $\Rightarrow \exists r_i > 0$ and $N_{r_i}(x) \subset G_i$ $(i=1,2,\ldots,m)$.
Take the minimum of the radius of each neighboor $r = \min(r_1, r_2, \ldots, r_m)$. then $N_r(x) \subset G_i$ $i=1,2,\ldots,m$; so that x is an interior point of $\bigcap_{k=1}^{m} G_k$ and thus $\bigcap_{k=1}^{m} G_k$ is open.

Now, ⓒ $\Rightarrow$ ⓓ since if $\bigcap_{k=1}^{m} G_k$ is open, then $\left(\bigcap_{k=1}^{m} G_k\right)^c$ is closed. But, by prev. theorem $\left(\bigcap_{k=1}^{m} G_k\right)^c = \bigcup_{k=1}^{m} G_k^c$, and since G_k is open $G_k^c = F_k$ is closed. therefore, $\bigcup_{k=1}^{m} F_k$ is closed. □

Let X be a metric space. Given $E \subset X$.

ⓐ Does there exists a <u>smallest</u> closed set F. such that $E \subset F$? YES!
(How far is the set from being closed).

ⓑ Does there exists a <u>largest</u> open set G such that $G \subset E$? YES!

<u>Pf</u> ⓐ Let $\mathfrak{F} = \{F : F \text{ is closed}, E \subset F\}$. Since $X \in \mathfrak{F}$, X being closed and $E \subset X$, then $\mathfrak{F} \neq \emptyset$. Moreover $\bigcap_{F \in \mathfrak{F}} F$ is closed (by previous proposition) and $E \subset \bigcap_{F \in \mathfrak{F}} F$. It remains to show that $\bigcap_{F \in \mathfrak{F}} F$ is the smallest closed set s.t $E \subset \bigcap_{F \in \mathfrak{F}} F$.
Suppose for a contradiction that $\exists F'$, a closed set s.t. $E \subset F'$ and $F' \subset \bigcap_{F \in \mathfrak{F}} F$.
then, since F' is closed and $E \subset F' \Rightarrow F' \in \mathfrak{F}$. But $F' \subset \bigcap_{F \in \mathfrak{F}} F$, so in particular $F' \subset F'$, a contradiction.

ⓑ Let $\mathcal{G} = \{G : G \text{ is open}, G \subset E\}$. Since $\emptyset \in \mathfrak{F}$, $\emptyset$ being open and $\emptyset \subset E$, then $\mathfrak{F} \neq \emptyset$. Moreover $\bigcup_{G \in \mathcal{G}} G$ is open (by previous proposition) and $\bigcup_{G \in \mathcal{G}} G \subset E$. It remains to show that $\bigcup_{G \in \mathcal{G}} G$ is the largest open set s.t. $\bigcup_{G \in \mathcal{G}} G \subset E$.
Suppose for a contradiction that $\exists G'$, an open set s.t. $G' \subset E$ and $\bigcup_{G \in \mathcal{G}} G \subset G'$.
then, since G' is open and $G' \subset E \Rightarrow G' \in \mathcal{G}$. But $\bigcup_{G \in \mathcal{G}} G \subset G'$, so in particular $G' \subset G'$, a contradiction.

Such smallest closed set such that E is contain in it is called <u>the closure</u> and is defined as $\bar{E} = E \cup E'$, where E' is the set of all limit points of E.

the properties of the closure are:

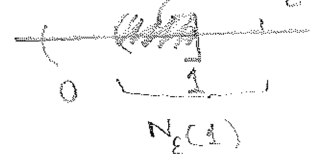
ⓐ $\bar{E}$ contains E ; ⓑ $\bar{E}$ is <u>closed</u> ⓒ $\bar{E}$ is the <u>smallest</u>.

theorem: (2.28). Let $E \subset X$ be s.t. $E \neq \emptyset$ and bounded above. Let $y = \sup E$. then, $y \in \bar{E}$. Pf: By definition, $\bar{E} = E \cup E'$. Suppose $y \in E$, then $y \in E \cup E' \Rightarrow y \in \bar{E}$. Otherwise, $y \notin E$. then we can show that $y \in E'$, i.e., that y is a limit point of E. let $r > 0$. $N_r(y) \setminus \{y\} \setminus E \neq \emptyset$. So that $y \in E' \Rightarrow y \in E \cup E' \Rightarrow y \in \bar{E}$.

Relative Spaces: Let (X, d) be a metric space. Let $A \subset X$. then (A, d) is also a metric space called the induced metric space.

Note that the shape of the neighboors may change. For example, let $A = (0, 1]$, inside of the reals with the same metric. then: $\to N_\varepsilon^A(1)$; so an open set here may look lik $(a, 1]$.

$$N_r^A(p) = \{q \in A : d(p,q) < r\}.$$



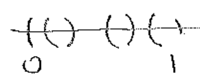
In general, open sets G^A in A are of the form: $G^A = G \cap A$, where G is an open set in X. Pf: $G^A = \bigcup_{p \in G^A} N_{r_p}^A(p) = \bigcup_{p \in G^A} (N_{r_p}(p) \cap A) = \left(\bigcup_{p \in G^A} N_{r_p}(p)\right) \cap A$, since the union of open sets is open.

COMPACT SETS: (X, d) a metric space. $E \subseteq X$.

Definition: the collection $\{G_\alpha\}$ of open sets form an open cover of E if

$$E \subset \bigcup_\alpha G_\alpha$$

EX: $(0,1)$ is an open cover of $(0,1)$



Taking succesive middle points. this form an open cover of $(0,1)$

Definition: $K \subset X$ is compact if every open cover of K contains a finite subcover. If $\{G_\alpha\}$ is an open cover of K then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ s.t.

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_m}.$$

Non-example: $E = (0,1]$ is not compact. Consider the open covering $\{(\frac{1}{n}, 2)\}_{n \in \mathbb{N}}$. then $E \subset \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 2)$; But any finite subcover won't include zero.

Example: Every finite set is compact. Let $K = \{x_1, x_2, \ldots, x_n\} \subseteq X$. Suppose that an open cover $\mathcal{U}$ is given. $\forall x_i \in K : \exists U_i \in \mathcal{U} : x_i \in U_i$. take $i = 1, \ldots, n$; the finite open sub cover is $K \subset \bigcup_{i=1}^n U_i$.

Compactness is an intrinsic Property: Suppose $K \subset Y \subset X$. then

K is compact in (Y, d) iff K is compact in (X, d).

$\Leftarrow$) Suppose K is compact in (X, d). Let $\{G_\alpha^Y\}$ be an open cover of K relative to Y, i.e., $K \subset \bigcup_\alpha G_\alpha^Y$; but by previous result, any open set of Y can be writen as $G_\alpha^Y = G_\alpha \cap Y$, where G_α is open in X.

therefore, $K \subset \bigcup_\alpha G_\alpha^Y = \bigcup_\alpha (G_\alpha \cap Y) = (\bigcup_\alpha G_\alpha) \cap Y \Rightarrow K \subset \bigcup_\alpha G_\alpha$.
But $\{G_\alpha\}$ is an open cover in X, so there exists $\alpha_1, \dots, \alpha_n$ such that $K \subset \bigcup_{\alpha=1}^{n} G_{\alpha_i}$. Finally, since $K \subset Y$ we have:
$K = K \cap Y \subset (\bigcup_{\alpha=1}^{n} G_{\alpha_i}) \cap Y = \bigcup_{\alpha=1}^{n} (G_{\alpha_i} \cap Y) = \bigcup_{\alpha=1}^{n} G_{\alpha_i}^Y \Rightarrow K \subset \bigcup_{\alpha=1}^{n} G_{\alpha_i}^Y$; where $G_{\alpha_i}^Y$ is open relative to Y. Hence, K is compact in (Y,d).

($\Rightarrow$) Suppose K is compact in (Y,d). Let $\{G_\alpha\}$ be an open cover for K relative to X, i.e., G_α is open in X. For every α, let $G_\alpha^Y = G_\alpha \cap Y$, so that G_α^Y is open relative to Y. then, $K = K \cap Y \subset (\bigcup_\alpha G_\alpha) \cap Y = \bigcup_\alpha (G_\alpha \cap Y) = \bigcup_\alpha G_\alpha^Y$. But K is compact in (Y,d), so, there exists $\alpha_1, \dots, \alpha_n$ s.t. $K \subset \bigcup_{\alpha=1}^{n} G_\alpha^Y$. But $G_\alpha^Y \subset G_\alpha$ for every α, so that $K \subset \bigcup_{\alpha=1}^{n} G_\alpha^Y \subset \bigcup_{\alpha=1}^{n} G_\alpha \Rightarrow K \subset \bigcup_{\alpha=1}^{n} G_\alpha$, so K is compact in (X,d)

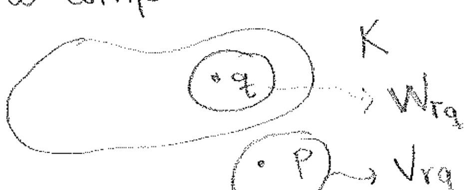
<u>Note</u>: we may sensibly speak of K as being compact without reference to the ambient space. In particular, this is always equivalent to K being compact as a metric space in its own right. (as a subset of itself, take $Y=K$).

<u>Properties of Compact sets</u>:

(1) Compact sets are closed.

<u>Pf</u>: Let K be a compact set. Let us prove that K^c is open.

Here p is fixed and q varies.



$\left.\begin{array}{l} q \in K \\ p \in K^c \end{array}\right\} \Rightarrow p \neq q \Rightarrow d(p,q) > 0$

Note that $W_{r_q}(q) \cap V_{r_q}(p) = \phi$, with $r_q = \frac{d(p,q)}{4}$. Since to each element $q \in K$ we associate $W_{r_q}(q)$, we can conclude that $\{W_{rq}\}$ form an open cover of K: $K \subset \bigcup_{q \in K} W_{r_q}(q)$. But K is compact, so there exists $q_1, q_2, \dots, q_n$ such that

$$K \subset \bigcup_{i=1}^{n} W_{r_{q_i}}(q_i)$$

Since $K \subset W_{rq_1} \cup \dots \cup W_{rq_n} \Rightarrow (W_{rq_1} \cup \dots \cup W_{rq_n})^c \subset K^c$; Also $V_{rq_i} \subset W_{rq_i}^c$ for $i=1,\dots,n$ then $V = V_{rq_1} \cap \dots \cap V_{rq_n} \subset W_{rq_1}^c \cap \dots \cap W_{rq_n}^c = (W_{rq_1} \cup \dots \cup W_{rq_n})^c \subset K^c \Rightarrow$ Since V is a neighborhood ... K is closed.

(2) $F \subset K$, F closed and K compact then F is compact.

Pf: Let $F \subset K$, F closed and K compact. Let $\{V_\alpha\}$ be an open cover of F. Since F is closed $\Rightarrow$ F^c is open. Note that $\Omega = \{V_\alpha\} \cup F^c$ is an open cover of K.

$\cup V_\alpha$ $\quad K \subset \left(\bigcup_\alpha V_\alpha\right) \cup F^c$; but K is compact, so there exists a finite subcove Ω' of Ω that cover K. Since $F \subset K$, Ω' will also cover F. Hence, Ω' is a finite open cover of F. $\Rightarrow$ F is compact.

(3) If F is closed and K is compact, then $F \cap K$ is compact

Pf: K compact $\Rightarrow$ K is closed. The intersection of closed sets is closed. So, $F \cap K$ is closed. Also, $F \cap K \subset K$; so apply previous theorem to conclude that: $F \cap K \subset K$ closed and K compact $\Rightarrow$ $F \cap K$ is compact.

theorem: If $\{K_\alpha\}$ are compact, and they have the finite intersection property (the intersection of every finite subcollection of $\{K_\alpha\}$ is non empty). Then,

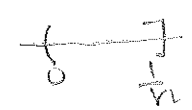
$$\bigcap_\alpha K_\alpha \neq \Phi.$$

Pf: By contradiction. Fix one of the K_α's say K_{α_1}. Assume that no point in K_{α_1} belongs to all the remaining K_α's. Then, $K_{\alpha_1} \subset \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$. Since K_α is compact, it is closed so K_α^c is open. Thus $\{K_\alpha^c\}$, $\alpha \neq \alpha_1$ is an open cover of K_{α_1}. But K_{α_1} is compact, so there exists $\alpha_2, \dots, \alpha_n$ such that $K_{\alpha_1} \subset \bigcup_{i=2}^n K_{\alpha_i}^c$, which means that $K_{\alpha_1}^c \supset \bigcap_{i=2}^n K_{\alpha_i}$. But then $\bigcap_{i=1}^n K_{\alpha_i} = \Phi$, a contradiction.

theorem: If E is an infinite subset of a compact set K, then E has a limit point in K.

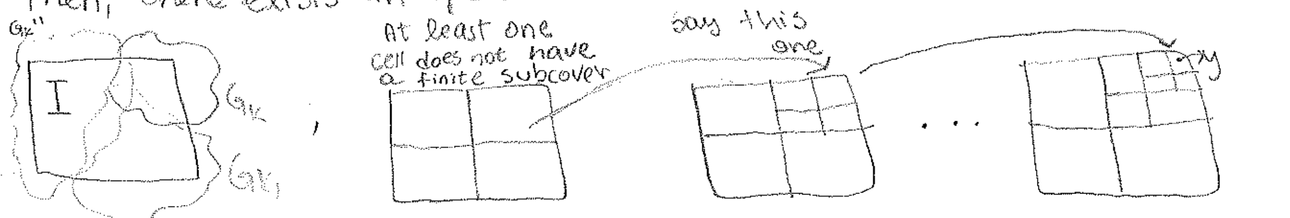
Pf: By contradiction. Suppose that E is an infinite subset of a compact set K and E has no limit points in K. Then, given $p \in E$, every $N_{r_p}(p)$ contains finitely many points of K. Moreover, $K \subset \bigcup_{p \in K} N_{r_p}(p)$ (in particular because $p \in N_{r_p}(p)$). But, K is compact, so there exists $p_1, \dots, p_n$ such that $K \subset \bigcup_{i=1}^n N_{r_{p_i}}(p_i)$. But $E \subset K$ $\Rightarrow$ $E \subset K \subset \bigcup_{i=1}^n N_{r_{p_i}}(p_i)$ $\Rightarrow$ $E \subset \bigcup_{i=1}^n N_{r_{p_i}}(p_i)$. But each $N_{r_{p_i}}(p_i)$ contains finitely many points; so $\bigcup_{i=1}^n N_{r_{p_i}}(p_i)$ is a finite set, so E most be finite, a contradiction with E being infinite.

<u>theorem</u>: If $\{I_n\}$ is a non-increasing sequence of closed intervals in $\mathbb{R}^1$, then $\bigcap_n I_n \neq \emptyset$.

<u>Pf</u>: this was a HW problem. A counter example would be , $\bigcap_n I_n = \emptyset$; but $0 \notin (0, \frac{1}{n}]$ for any n.

<u>theorem</u>: Every closed k-cell is compact.

<u>Pf</u>: For $n=2$. By contradiction. Suppose there exist a 2-cell, $I \in \mathbb{R}^2$, that is not compact. then, there exists an open cover $\{G_\alpha\}$ of I which admits no finite subcover.



continue this proces ad infinitum.

So we have a sequene of decreasing k-cells. By previous theorem (2.39), we have that $\bigcap_{\substack{\text{decreasing}\\k\text{-cells}}} I_k \neq \emptyset$. Let $y \in \bigcap_{\substack{\text{decreasing}\\k\text{-cells}}} I_k$. then $y \in G_{\alpha_y}$, , so the cell admits a finite subcover, a contradiction. (the cell only contains y, since the length of decreasing I_k goes to zero as $k \to \infty$)

this proves works on $\mathbb{R}^n$, just divide into 2^n parallelepipes.

<u>THEOREM</u>: Let $E \subset \mathbb{R}^k$. then the following conditions are equivalent:

(a) E is closed and bounded
(b) E is compact
(c) Every infinite subset of E has a limit point in E.

Note that this is true only in $\mathbb{R}^k$. <u>A counterexample</u> is X = infinite set, d = 0-1 metric (if $x=y$, $d(x,y)=0$, otherwise $d(x,y)=1$). X is closed and bounded but not compact.

<u>Pf</u>: (a) $\Rightarrow$ (b). Suppose $E \subset \mathbb{R}^k$ is closed and bounded. then $E \subset I$ for some k-cell. I. We showed that k-cells are compact. $E \subset I$, E closed, I compact $\Rightarrow$ E compact.

(b) $\Rightarrow$ (c). Suppose $E \subset \mathbb{R}^k$ is compact. Let $F \subset E$ be an infinite subset of E. By theorem 2.37, since $F \subset E$, F infinite and E compact, then F has a limit point in E.

(c) $\Rightarrow$ (a) Suppose $E \subset \mathbb{R}^k$ such that every infinite subset of E has a limit point in E. Let us prove closed and bounded separately.

(i) By contraposition: Suppose that E is not bounded. then, there exists points x_n in E such that $|x_n| > n$ $(n=1,2,3,\dots)$. let $S = \{x_n \mid x_n \in E$ and $|x_n| > n, n=1,2,3,\dots\}$. Clearly S is infinite. Moreover, S has no limit points in $\mathbb{R}^k$. Suppose it does. then, there exists $p \in S$ such that $\forall r > 0$: $N_r(p) \setminus \{p\} \cap \mathbb{R}^k \neq \emptyset$. Let $y \in N_r(p) \setminus \{p\} \cap \mathbb{R}^k$. then, $y \in N_r(p)$, $y \neq p$, $y \in \mathbb{R}^k$. But $p \in S$, $|p| > n$, $(n=1,2,3,\dots)$. $d(p,y) < r$, a contradiction.
So S has no limit points in $\mathbb{R}^k$ $\Rightarrow$ S has no limit points in E.

(ii) By contraposition: Suppose that E is not closed.

CANTOR SETS:

$C_0 = [0,1]$ — $2^0 = 1$ closed interval

$C_1 = [0, 1/3] \cup [2/3, 1]$ — $2^1 = 2$ closed intervals

$C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [\frac{8}{9}, 1]$ — $2^2 = 4$ closed intervals

length of intervals in $C_n = \frac{1}{3^n}$; # of intervals in $C_n = 2^n$.

the Cantor set is $C = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$, since $0 \in C$. In general all endpoints are in C.

to represent elements of C, use ternary expansion. $.x_1 x_2 x_3 \dots ; x_n = 0,1,2, \Rightarrow$
$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$. For example: $\frac{1}{3} = 0$ and all twos:

$$\frac{2}{3} = \frac{1}{3} + \sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3} + 2\,\frac{\frac{1}{3^2}}{\frac{2}{3}} = \frac{1}{3} + 2\cdot\frac{1}{3\cdot 2} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Properties of CANTOR SET:

(I) Each point $x \in X$, has a ternary expansion; $x = .x_1 x_2 x_3 \dots$, where $x_1 \neq 1, x_2 \neq 1, x_3 \neq 1, \dots, x_n \neq 1$ $\forall n$. Hence, there are only 0 or 2 in the expansion. For example: A 0 indicates being to the left and a 2 indicates being to the right.
$1/3 = 0.0222\dots$; $7/9 = 0.20222\dots$

(II) the complement of C, i.e., $[0,1] \setminus C$ is open. Is the countable union of open intervals, and if we sum the length of all those intervals we get:

$$2^0\frac{1}{3} + 2^1\frac{1}{3^2} + 2^2\frac{1}{3^3} + \dots = \sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{2^n}{3^n} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3}\left(\frac{1}{\frac{1}{3}}\right) = \frac{1}{3}\cdot 3 = \boxed{1}$$

(III) Points in C include all the endpoints of the C_n's.

(IV) How many points are there in (III) Countably many (Countable union of finite sets $\bigcup_{i=1}^{\infty}$ endpoints in C_i). How many points are there in (I)? i.e., how many points are there in the Cantor set? Uncountable many, this follows from the result that 2-valued sequences are uncountable. A point in C is a sequence $.x_1x_2x_3\cdots$) with no ones. For example $\frac{1}{4} \in C$, since

$\frac{1}{4} = 0.020202\cdots$, Note that $\frac{1}{4}$ is not an endpoint.

(V) C is compact, it contains no interval.

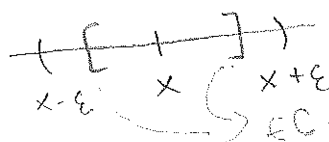
(i) C is clearly bounded since it is contained in $[0,1]$.

(ii) C is closed. Each C_n is a finite union of closed sets; so C_n is closed the Cantor set is the intersection of all of these sets. the intersection of closed sets is closed, so C is closed

(i) and (ii) $\Rightarrow$ C is compact.

(VI) C is perfect. We already showed that C is closed. It remains to show that $x \in C \Rightarrow x$ is a limit point of C (i.e, C has no isolated points)

Let $x \in C$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be sufficiently large s.t. $\frac{1}{3^n} < \varepsilon$. Consider $N_\varepsilon(x)$



So, there exists $y \in N_\varepsilon(x)$, $y \neq x$ (you can choose the endpoint of C). s.t $N_\varepsilon(x) \setminus \{x\} \cap C \neq \emptyset$.

x is a limit point of C.

(VII) C is symmetric about $\frac{1}{2}$; $x \in C \Leftrightarrow 1-x \in C$.

To prove this, let $x \in C$. write x in ternary expansion:

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$$; by definition of being in C, $x_n \neq 1$ for all n.

Now, write 1 is Ternary expansion $1 = \sum_{n=1}^{\infty} \frac{2}{3^n}$. Take the difference:

$$1 - x = \sum_{n=1}^{\infty} \frac{2}{3^n} - \sum_{n=1}^{\infty} \frac{x_n}{3^n} = \sum_{n=1}^{\infty} \frac{2-x_n}{3^n}$$; since $x_n \neq 1$, $2 - x_n = \begin{cases} 2 & \text{if } x_n = 0 \\ 0 & \text{if } x_n = 2 \end{cases}$

$\Leftrightarrow$ $2 - x_n \neq 1$ for all n $\Leftrightarrow$ $1-x$ contains no ones in its ternary expansion

$\Leftrightarrow$ $1-x \in C$.

Other related properties are:

$x \in C \Rightarrow \frac{x}{3} \in C$; If $x < \frac{1}{3}$ and $x \in C \Rightarrow 3x \in C$.

All of these properties can be proved using the Ternary expansion

(VIII) Each point in $[0,1]$ is the mid point of not neccesarily distinct points in C, i.e., $x \in [0,1] \Rightarrow x = \frac{y+z}{2}$; $y, z \in C$.
We are going to construct numbers y, z that satisfies the above condition.
First, let $x \in [0,1]$. Consider its ternary expansion: $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$.
Construct sequences y_n, z_n as follow:

$$y_n = \begin{cases} x_n & \text{if } x_n = 0, 2 \\ 2 & \text{if } x_n = 1 \end{cases} \quad ; \text{ and } \quad z_n = \begin{cases} x_n & \text{if } x_n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the numbers: $y = \sum_{n=1}^{\infty} \frac{y_n}{3^n} \in C$; $z = \sum_{n=1}^{\infty} \frac{z_n}{3^n} \in C$. take the sum:

$$y + z = \sum_{n=1}^{\infty} \frac{y_n}{3^n} + \sum_{n=1}^{\infty} \frac{z_n}{3^n} = \sum_{n=1}^{\infty} \frac{y_n + z_n}{3^n},$$

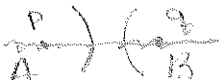
If $x_n = 1 \Rightarrow y_n + z_n = 2 + 0 = 2$
If $x_n = 0 \Rightarrow y_n + z_n = 0 + 0 = 0$
If $x_n = 2 \Rightarrow y_n + z_n = 2 + 2 = 4 = 1$

$y + z = 2x \Rightarrow x = \frac{y+z}{2}$.

(IX) $C - C = [-1, 1]$ and $C + C = [0, 2]$ moreover, the statements are equivalent.
Let us prove that $C + C = [0, 2]$.
($\subseteq$) Clearly, since C is bounded by $[0,1]$, $C + C \subset [0, 2]$.
($\supseteq$) Let $x \in [0, 2] \Rightarrow \frac{x}{2} \in [0,1] \underset{\text{by (VIII)}}{\Rightarrow} \frac{x}{2} = \frac{y+z}{2}$; $y, z \in C$. $\Rightarrow x = y + z$

<u>Connected Sets</u>: Let A, B be sets in a metric space. $A \cap B = \phi$ means A, B are disjoint sets. this is not the same as the notion of separated: $\bar{A}$ = closure of A
<u>Separated</u>: two sets A, B are separated if $A \cap \bar{B} = \bar{A} \cap B = \phi$
<u>Example</u>: let $A = (0,1)$, $B = (1,2)$. then $(0,1) \cap [1,2] = [0,1] \cap (1,2) = \phi$. So, A, B are separated
<u>Example</u>: Let $A = [0,1)$ and $B = (1,2]$ then $[0,1) \cap [1,2] = \phi = [0,1] \cap (1,2] = \phi$
SEPARATED sets $\Rightarrow$ disjoint. but not the converse:
Example: Let $A = (0,1]$; $B = (1,2)$. then $(0,1] \cap [1,2] \neq \phi \Rightarrow A, B$ not separated
But $A \cap B = (0,1] \cap (1,2) = \phi \Rightarrow A, B$ are disjoint.

<u>Def</u>: $E \subset X$; a subset of a metric space is <u>connected</u> if E is <u>not</u> the union of two nonempty separated sets.
<u>Examples</u>: this types of sets $[\ ]$, $(\ )$, $[\ )$, $(\ ]$ are connected.
<u>Non examples</u>: CANTOR SET, $\mathbb{Q}$ and the set of irrationals.
<u>Intrinsic Property</u>: Just like for being compact, the notion of connected is
<u>INTRINSIC</u>



; A, B open in E and separated
$d(p,q) > r_p + r_q$

Analysis I · Enrique Areyan - Fall 2013 (13)

CHAPTER 3: Sequences.

terms of the sequence

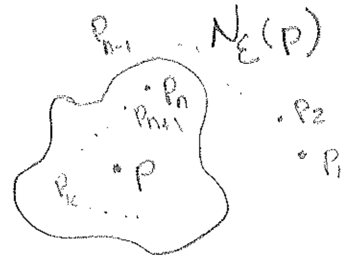
Let (X,d) be a metric space.

Def: A function $f: \mathbb{N} \to X$, is called a sequence $f(1), f(2), \ldots$ the sequence is usually denoted as $\{p_n\}$.

the range of the sequence is just the range of f. this might not be the whole space. there might repeated terms.

Def: Convergent Sequences. Given a sequence $\{p_n\} \subset X$ and a point $p \in X$, we say that $\{p_n\}$ converges to p and we write:

$$\begin{cases} \lim_{n\to\infty} p_n = p \\ \lim_n p_n = p \end{cases} \qquad \begin{cases} p_n \to p \text{ as } n \to \infty \\ p_n \to p \end{cases}$$



Provided that:

$\forall \varepsilon > 0 : \exists N : \forall n \geq N : d(p_n, p) < \varepsilon$.

Def: If $\{p_n\}$ does not converge, it is said to diverge.

$\exists \varepsilon > 0 : \forall N : \exists n \geq N : d(p_n, p) \geq \varepsilon$.

Properties of sequences: let $\{p_n\}$ be a sequence in a metric space (X,d)

(a) $p_n \to p \Leftrightarrow$ Every neighborhood of p contains all but finitely many p_n's.

Pf: $(\Rightarrow)$ Suppose $p_n \to p$. Let V be a neighborhood of p. By definition; $\forall \varepsilon > 0 : \exists N : \forall n \geq N : d(p, p_n) < \varepsilon$. Let $\varepsilon > 0$. Consider $V_\varepsilon(p)$. Choose N and $n \geq N$ s.t. $d(p, p_n) < \varepsilon$. then $p_n \in V_\varepsilon(p)$. Hence, $V_\varepsilon(p)$ contains all but finitely many p_n, i.e., those for which $n < N$.

$(\Leftarrow)$ Suppose every neighboord of p contains all but finitely many p_n's. Let $\varepsilon > 0$. $N_\varepsilon(p)$ contains all but finitely many p_n's. Hence, $\exists N$ s.t. $\forall n \geq N$ $d(p, p_n) < \varepsilon$, i.e., $p_n \in N_\varepsilon(p)$. this is the same as $p_n \to p$.

(b) If $p_n \to p$ and $p_n \to p'$, then $p = p'$. Limits are unique.

Pf: Suppose that $p_n \to p$ and $p_n \to p'$ and $p \neq p'$. (P_f by contradiction).

By definition. given $\varepsilon > 0$ $\begin{cases} \exists N : \forall n \geq N : d(p_n, p) < \varepsilon. \\ \exists N' : \forall n \geq N' : d(p_n, p') < \varepsilon. \end{cases}$

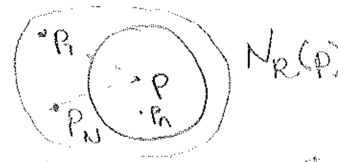
Let $\varepsilon = \frac{d(p,p')}{2} > 0$. Pick $n \geq \max(N, N')$. then: $d(p_n, p) < \varepsilon = \frac{d(p,p')}{2} \Rightarrow d(p_n, p) < \frac{d(p,p')}{2}$

likewise $d(p_n, p') < \varepsilon = \frac{d(p,p')}{2} \Rightarrow d(p_n, p') < \frac{d(p,p')}{2}$. But then

$d(p,p') \leq d(p,p_n) + d(p_n,p') < \frac{d(p,p')}{2} + \frac{d(p,p')}{2} = d(p,p')$

$\Rightarrow d(p,p') < d(p,p')$, a contradiction. Therefore $p = p'$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded, i.e., $\{p_n\} \subset E \subset X$.



<u>Pf</u>: Suppose that $p_n \to p$. Let $\varepsilon > 0$. Then, there exists N s.t. $n \geq N \Rightarrow d(p_n,p) < \varepsilon$. Choose $\varepsilon = 1$. Then $d(p_n,p) < 1$. Now, measure the distance of all points up to N (this are finitely many, so we can do this). $d(p,p_1), d(p,p_2), \dots, d(p,p_N)$. Now set R to be

$R = \max\{d(p,p_1), d(p,p_2), \dots, d(p,p_N), 1\}$. → the choice of 1 here is arbitrary, any number > 0 will work.

then $\{p_n\} \subseteq N_R(p)$.

(d) <u>If</u> $E \subset X$ and if p is a limit point of E <u>then</u> there exists a sequence $\{p_n\} \subset E$ such that $p_n \to p$.

<u>Pf</u>: Let $E \subset X$ and p a limit point of E. Construct the sequence $\{p_n\}$ as follow.
Since p is a limit point: $\forall r > 0$: $N_r(p) \setminus \{p\} \cap E \neq \emptyset$.
Let $p' \in N_r(p) \setminus \{p\} \cap E \Rightarrow p' \in N_r(p)$ and $p' \neq p$ and $p' \in E$.
Now, let r vary with $n \in \mathbb{N}$ s.t. $r = r_n = \frac{1}{n}$. Then we can always find a point $p_n \in N_r(p) \setminus \{p\} \cap E$. The sequence we want is $\{p_n\}$. This sequence converges to p.
Let $\varepsilon > 0$. Choose $N > \frac{1}{\varepsilon}$, and $n \geq N$. Then: $d(p,p_n) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon \Rightarrow p_n \to p$.

RELATION BETWEEN convergence and algebraic operations. These operations may not be defined in arbitrary metric spaces, so let $X = \mathbb{R}$. (or $\mathbb{R}^k$).
<u>Properties</u>: Let $\{s_n\}$ and $\{t_n\}$ be real valued sequences s.t. $s_n \to s$ and $t_n \to t$.

(a) $s_n + t_n \to s + t$

<u>Pf</u>: By def. Given $\varepsilon > 0$: $\begin{cases} \exists N : \forall n \geq N : |s_n - s| < \varepsilon/2 < \varepsilon \\ \exists N' : \forall n \geq N' : |t_n - t| < \varepsilon/2 < \varepsilon \end{cases}$

Let $\varepsilon > 0$. Choose N, N' and $n \geq \max(N,N')$, such that the above holds. Then,

$$|(s_n+t_n) - (s+t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow |(s_n+t_n) - (s+t)| < \varepsilon \Rightarrow s_n + t_n \to s + t$ as $n \to \infty$.

(b) $cs_n \to cs$ and $c + s_n \to c + s$; for any number c.

<u>Pf</u>: By def. Given $\varepsilon > 0$: $\exists N : \forall n \geq N : |s_n - s| < \varepsilon$. Let $\varepsilon > 0$, pick N, $n \geq N$ so be such that that the above hold. Then: $|cs_n - cs| = |c(s_n - s)| = |c||s_n - s| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon$

CASE $c \neq 0$

Note $|s_n - s| < \varepsilon \Rightarrow |s_n - s| < \frac{\varepsilon}{|c|}$

CASE $c = 0$ $\quad 0 \cdot s_n \to 0 \cdot s$ trivially.

SECOND CASE: $c+s_n \to c+s$.

Let $\varepsilon > 0$. choose N and $n \geq N$ s.t. $|s_n - s| < \varepsilon$

But then: $|(c+s_n)-(c+s)| = |s_n - s| < \varepsilon \Rightarrow c+s_n \to c+s$.

(c) $s_n t_n \to st$.

Pf:

(d) $\frac{1}{s_n} \to \frac{1}{s}$ provided that $s_n \neq 0 (n=1,2,\dots)$ and $s \neq 0$

Pf: $\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{|s_n - s|}{|s_n||s|} \leq \frac{1}{A'}\frac{|s_n-s|}{|s|}$; $A' \leq |s_n| < A$ (bounded)

$\leq \frac{\varepsilon}{|s|} < \varepsilon$

THEOREM: We can use the above proved properties in $\mathbb{R}$ to extended to the case $\mathbb{R}^k$. note that for property (c) we would use the inner product

$(x_1, \dots, x_k) = \mathbf{X}$; $\mathbf{X}_n \to \mathbf{X}$ iff $x_1^n \to x_1, \dots, x_k^n \to x_k$

So all properties hold: if $\mathbf{X}_n \to \mathbf{X} \wedge \mathbf{Y}_n \to \mathbf{Y}$: $\mathbf{X}_n + \mathbf{Y}_n \to \mathbf{X}+\mathbf{Y}$; $\mathbf{X}_n \cdot \mathbf{Y}_n \to \mathbf{X}\cdot\mathbf{Y}$; $c\mathbf{X}_n \to c\mathbf{X}$

---

the class of divergent sequences is very large and important to deal with it, we introduce the notion of subsequence.

Definition: Given a sequence $\{p_n\}$, let $n_1 < n_2 < \dots n_k < \dots$, the sequence $\{p_{n_k}\}$ is called a subsequence of $\{p_n\}$. Moreover,

$\{p_n\} \to p$ iff every subsequence of $\{p_n\}$ converges to p.

THEOREM: Let (X,d) be a compact metric space.

(a) $\{p_n\} \subset X$ then $\{p_n\}$ has a convergent subsequence

(b) (H-B). Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence

Pf: ⓐ Two cases:

(i) range of $\{p_n\}$ is finite. then we can list the values of the range as boxes: ☐ ☐ .... ☐ ; at least one box will have infinitely many terms. (by Pigeonhole principle).
Pick $\{n_k\}$ s.t P_{n_k} are all in such a box. then $P_{n_k} \to$ the value of the box so $\{P_{n_k}\}$ converges (for case (i) we do not need compactness).

(ii) range of $\{p_n\}$ is infinite. Since $\{p_n\} \subset X$ and X is compact, and the range is infinite, by theorem 2.37, $\{P_n\}$ has a limit point in X. Call it p.

$N_{\frac{1}{k}}(p)$, $k=1,2,3,\dots$

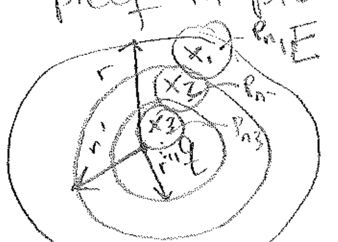
By theorem 2.20, since p is a limit point, every neiborhood of p contains infinitely many points of $\{p_n\}$ (range of $\{p_n\}$). So choose neiborhoods with radius $r=\frac{1}{k}$; $k=1,2,3,\dots$ pick a point in each neiborhood $P_{n_k} \in N_{\frac{1}{k}}(p)$;
then $\{P_{n_k}\}$ is a subsequence and in fact $\{P_{n_k}\} \to p$.

ⓑ Let $\{p_n\}$ be a bounded sequence in $\mathbb{R}^k$. Since $\{p_n\} \subset \mathbb{R}^k$ and it is bounded, we can find a k-cell I, s.t. $\{p_n\} \subset I$.

I ; So I is a closed and bounded set. therefore is compact. So $\{P_n\}$ lies in a compact set and per proof of ⓐ; $\{p_n\}$ has a convergent subsequence.

THEOREM: the subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X. In other words. let $\{p_n\}$ be a seq. Define $E = \{p \in X : \exists$ a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ that converges to $p\}$ then E is closed. (Question, if $\{p_n\} \to p$ then every subsequence $\to p$. So if $\{p_n\} \to p$ ... $= \{p\}$.).

Pf. Let q be a limit point of E. Want to show $q \in E$. $\Leftrightarrow$ q is a subsequential limit point of $E \Leftrightarrow \exists \{P_{n_k}\}$ subsequence of $\{P_n\}$ such that $P_{n_k} \to q$. Idea for proof in pictures:



SHRINK $r, r', r'', \dots$ Since q is a limit point, $\forall r > 0\ \exists x_n$ s.t. $x_n \in E$, $x_n \in N_r(q)$, $x \neq q$; but each $x_n \in E \Rightarrow x_n$ is a subsequential limit point
$\Rightarrow \exists P_{n_k} \in N_{\text{any}}(x)$

therefore, the sequence $P_{n_1}, P_{n_2}, \ldots, P_{n_k}, \ldots$ converges to q and is a subsequence of P_n. Hence, $q \in E$. So E contains all its limit points. E is closed.

CAUCHY SEQUENCES:

Definition: A sequence $\{p_n\}$ in a metric space (X,d) is said to be Cauchy or a Cauchy sequence if:

$$\boxed{\forall \varepsilon > 0 : \exists N : \forall n,m \geq N : d(p_n,p_m) < \varepsilon}$$

Definition: A metric space such that all cauchy sequences converges is called COMPLETE.

Definition: Let E be a non-empty subset of a metric space X. Let $S = \{ d(p,q) \mid p,q \in E \}$. then $\sup S$ = diameter of E. $= \text{diam}(E)$

Proposition: Let $\{p_n\}$ be a sequence in X. Let $E_N = \{p_N, p_{N+1}, p_{N+2}, \ldots\}$

$\{p_n\}$ is Cauchy $\iff \lim_{N \to \infty} \text{diam}(E_N) = 0$.

Pf: $(\Rightarrow)$ let $\{p_n\}$ be Cauchy. WANT to prove $\{\text{diam}(E_n)\} \to 0$

$(\Leftarrow)$ Suppose $\lim_{m \to \infty} \text{diam}(E_m) = 0$. WANT to prove $\{p_n\}$ is cauchy.

By hypothesis, $\exists$ a positive integer K s.t. if $m > K \Rightarrow \text{diam}(E_m) < \varepsilon$.

$\Rightarrow \sup\{ d(u,v) : u \in E_m \wedge v \in E_m\} < \varepsilon$.

In particular, for $n, j > m$, we can write $n = m + x$ and $j = m + y$, for positive integers x and y. then $(P_n \in E_m, P_j \in E_M)$.

$$d(P_n, P_j) \leq \sup\{d(u,v) : u \in E_m \wedge v \in E_m\} < \varepsilon.$$

$\Rightarrow \{p_n\}$ is Cauchy.

General Strategy to show that $x=y$; $x,y \in \mathbb{R}$:

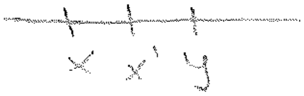
You can show ① $x \le y$ and ② $x \ge y$ $\Rightarrow x=y$. OR, you can give yourself more room and show: Given $\varepsilon_1>0$ and $\varepsilon_2>0$: $x-\varepsilon_1 < y < x+\varepsilon_2$. then $x=y$ because:

ⓐ: If Given $\varepsilon_1>0$ we have $x-\varepsilon_1<y$, suppose $x>y$. then

Pick $\varepsilon = \frac{x-y}{2}$; then $x' = x-\varepsilon > y$. But by hypothesis for any $\varepsilon_1>0$; In particular for ε we have $x-\varepsilon<y$ $\rightsquigarrow$ A contradiction.

Hence, $x \le y$.

ⓑ: If Given $\varepsilon_2>0$ we have $y<x+\varepsilon_2$, suppose $x<y$. then



Pick $\varepsilon = \frac{y-x}{2}$; then $x' = x+\varepsilon < y$. But by hypothesis, for any $\varepsilon_2>0$; In particular for ε we have $x+\varepsilon>y$ $\rightsquigarrow$ A contradiction.

Hence, $x \ge y$.

By ⓐ and ⓑ: $x\le y$ and $x \ge y$, then $\boxed{x=y}$

THEOREM 3.10:

(a) $diam(E) = diam(\bar{E})$

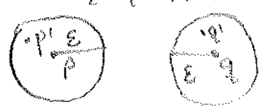
(b) If K_n is a decreasing sequence of compact sets $K_{n+1} \subset K_n$ $(n=1,2,3,\dots)$ and $\lim_{n\to\infty} diam\, K_n = 0$ then $\bigcap_{n=1}^{\infty} K_n = \{p\}$, a single point

Pf: ⓐ By definition, $E \subset \bar{E} \Rightarrow diam(E) \le diam(\bar{E})$.

Now, we want to prove $diam(\bar{E}) \le diam(E)$ to conclude $diam(E) = diam(\bar{E})$

However, we will use the strategy outlined above and show, for an arbitrary ε that $diam(\bar{E}) \le diam(E) + \varepsilon$.

Let $\varepsilon>0$. Pick $p,q \in \bar{E}$ so that $p \ne q$, $\varepsilon < \frac{d(p,q)}{3}$. By definition of $\bar{E}$, every point in $\bar{E}$ is a limit point of E, in particular p and q. Hence, $\forall r_1>0$ and $\forall r_2>0$ $N_{r_1}(p)\setminus\{p\} \cap E \ne \emptyset$ and $N_{r_2}(q)\setminus\{q\} \cap E \ne \emptyset$ Let $r_1=r_2=\varepsilon$. Pick $p' \in N_\varepsilon(p)\setminus\{p\} \cap E$ and $q' \in N_\varepsilon(q)\setminus\{q\}\cap E$.



: $d(p,q) \le d(p,p') + d(p',q') + d(q',q)$

$\le \varepsilon + d(p',q') + \varepsilon$

$\le 2\varepsilon + d(p',q')$

$\le 2\varepsilon + diam(E)$

But $p',q' \in E$, so $d(p',q') \le diam(E)$

So $d(p,q) \le 2\varepsilon + diam(E)$; but $p,q \in \bar{E}$ are arbitrary so $2\varepsilon + diam(E)$ is an upper bound for its distance so it has to be bigger than the least upper bound: $diam(\bar{E})$

$\Rightarrow$ $diam(\bar{E}) \le 2\varepsilon + diam(E)$ ...

ⓑ By contradiction; suppose there are two points $p, q \in \bigcap K_n$. Since $p \neq q$, $d(p,q) > 0$. Let $\eta = d(p,q) > 0$. Pick N s.t. $\text{diam}(K_N) < \frac{\eta}{2}$. Since p, q belong to every K_i, they belong to K_N.

$\eta = d(p,q) \leq \text{diam}(K_N) < \frac{\eta}{2} \Rightarrow \eta < \frac{\eta}{2}$; a contradiction.

THEOREM 3.11: (X,d) a metric space.

ⓐ If $\{p_n\}$ is convergent in (X,d) then $\{p_n\}$ is Cauchy

ⓑ If (X,d) is a compact metric space and $\{p_n\}$ is Cauchy then $\{p_n\}$ converges to some $p \in X$.

ⓒ In $\mathbb{R}^k$ every Cauchy sequence converges

Pf: ⓐ Let $p_n \to p$. Let $\varepsilon > 0$. Pick N s.t. $d(p, p_k) < \varepsilon$, whenever $k \geq N$. whenever $n \geq N$ and $m \geq N$.

Now, $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

Hence, $\{p_n\}$ is Cauchy.

ⓑ Let $\{p_n\}$ be a Cauchy sequence in a compact metric space. For $N = 1, 2, 3, \ldots$, Let $E_n = \{p_N, p_{N+1}, p_{N+2}, \ldots\}$. We have the following about $\overline{E_n}$:

ⓘ $\overline{E_{n+1}} \subset \overline{E_n}$

ⓘⓘ $\overline{E_n}$ is compact for every n, since $\overline{E_n}$ is closed and subset of a compact space.

ⓘⓘⓘ $\lim_{n\to\infty} \text{diam}(\overline{E_n}) = 0$

By previous theorem, ⓘ, ⓘⓘ & ⓘⓘⓘ $\Rightarrow \bigcap \overline{E_n} = \{p\}$. claim: $p_n \to p$. Pf: Let $\varepsilon > 0$. by (iii) $\exists M$ s.t. $\text{diam}(\overline{E_n}) < \varepsilon \ \forall n \geq M$. Since p belongs to all $\overline{E_N}$ $\Rightarrow d(p,q) < \varepsilon \ \forall q \in \overline{E_n} \Rightarrow \forall q \in E_n$. Relabeling $q = p_n \in E_n \Rightarrow d(p, p_n) < \varepsilon \ \forall n \geq M$. $p_n \to p$.

ⓒ $\boxed{\text{This is QUESTION 2 HW7}}$ this follows from (b): we proved that a Cauchy sequence is bounded. In $\mathbb{R}$, chose a k-cell, i.e., a closed interval $[a,b]$ that contains the sequence this k-cell is compact, and $\{p_n\}$ is contained in it. By ⓑ $\{p_n\}$ converges.

Lim inf and Lim sup.

Motivation: most sequences are "divergent", but we can still study them.

Consider $0, 1, 0, 1, 0, \ldots$ this is a bounded sequence with 2 convergent subsequences

$p_1 = p_3 = \ldots = 0$; $p_2 = p_4 = \ldots = 1$.

this sequence has a "largest" and "smallest" convergent subsequence. By largest we mean that converges to the largest value.

The setting here is going to be $\mathbb{R}$, because sup is defined here. Also,
$\{p_n\} \subset \mathbb{R}$ and $\{p_n\}$ bounded $\Rightarrow$ $\{p_n\}$ has a convergent subsequence.
$\{p_n\} \subset \mathbb{R}$, p_n increasing, bounded above then $\lim_n p_n = \sup_n p_n$.

Note that If $\{p_n\}$ is bounded then $\{-p_n\}$ is also bounded
If you can find the largest convergent subsequence of $\{p_n\}$ then you can do the same for $\{-p_n\}$.
If $\{p_{n_e}\}$ be the largest subsequence of $\{-p_n\}$ then $-\{p_{n_e}\}$ is the smallest subsequence of $\{p_n\}$.

Pf: Let $\{p_{n_e}\}$ be the largest subsequence of $\{-p_n\}$: $\cdot\, p_{n_e} \to \ell$; ℓ the largest converging value. Claim: $-\{p_{n_e}\}$ is s.t. $-p_{n_e} \to s$; s is the smallest value. Let $\varepsilon > 0$ $\exists N$ s.t.
$|-p_{n_e} - \ell| < \varepsilon \quad \forall n_e \geq N$.

How to look for the larger subsequence of a sequence? Recall here that $\{p_n\} \subset \mathbb{R}$ and $\{p_n\}$ is bounded.

$$\sup\{p_1, p_2, p_3, p_4, \dots\} \underset{\text{drop first term}}{\geq} \sup\{p_2, p_3, p_4, \dots\} \underset{\text{drop second term}}{\geq} \sup\{p_3, p_4, \dots\} \geq \dots \underset{\text{drop } n-1 \text{ term}}{\geq} \sup\{p_n, p_{n+1}, \dots\} \geq \dots$$

Since $\{p_n\}$ is a non-empty, bounded set of real numbers, we can look at: (by completeness).
$$\sup\{p_n : n \geq k\}, \quad k = 1, 2, 3, \dots$$

Definition: $\inf_k \sup\{p_n : n \geq k\}$, $k = 1, 2, 3$ $= \lim_k \sup\{p_n : n \geq k\}$.
this quantity is called $\limsup p_n$.
Equivalently: $\inf\{p_1, \dots\} \leq \inf\{p_2, p_3, \dots\} \leq \inf\{p_3, \dots\} \leq \dots \leq \inf\{p_n, p_{n+1}, \dots\} \leq \dots$
$$\sup_k \inf\{p_n : n \geq k\}, \ k = 1, 2, 3 \ = \lim_k \inf\{p_n : n \geq k\}$$

Proposition: $\liminf_n \{p_n\} = -\limsup_n \{-p_n\}$

To prove this, let us first prove:
If A is a bounded (bounded below?) set of real numbers, $A \neq \emptyset$, then $\inf A = -\sup(-A)$.

Pf: (i) $\inf(A) \leq -\sup(-A)$. Let $p \in A$. $\inf(A) \leq p \Rightarrow -\inf(A) \geq -p$, so $-\inf(A)$ is an upper bound of $-A \Rightarrow \sup(-A) \leq -\inf(A) \Rightarrow \boxed{\inf(A) \leq -\sup(-A)}$

(ii) $-\sup(-A) \leq \inf(A)$. Let $-p \in -A$. $-p \leq \sup(-A) \Rightarrow p \geq -\sup(-A)$, so $-\sup(-A)$ is a lower bound of $A \Rightarrow \boxed{-\sup(-A) \leq \inf(A)}$

(i) & (ii) $\Rightarrow \inf(A) = -\sup(-A)$.

Pf of: $\lim_n \inf \{p_n\} = -\lim_n \sup \{-p_n\}$

By definition $\lim_n \inf \{p_n\} = \sup_k \{\inf\{p_n : n \geq k\}\}$

$= -\inf_k \{-\inf\{p_n : n \geq k\}\}$

$= -\inf_k \{\sup\{-p_n : n \geq k\}\}$

$= -\lim_n \sup\{-p_n\}$ //

We proved that $\boxed{\inf A = -\sup(-A)}$

$\Rightarrow -\inf A = \sup(-A)$

$\Rightarrow -\inf(-B) = \sup(B)$.

by definition.

Proposition: Let $\{p_n\} \subset \mathbb{R}$ be bounded. Then

$\lim_n \inf\{p_n\} \leq \lim_n \sup\{p_n\}$.

Pf: Let $a = \lim_n \inf\{p_n\}$. By definition, $a = \sup_k \{\inf\{p_n : n \geq k\}\}$. $a \geq \inf\{p_n : n \geq k\}$ for all k.

Let $b = \lim_n \sup\{p_n\}$. By definition, $b = \inf_k \{\sup\{p_n : n \geq k\}\}$. $b \leq \sup\{p_n : n \geq k\}$ for all k.

Let $E_k = \{p_n : n \geq k\}$. Let $I_k = \inf\{E_k\}$ and $S_k = \sup\{E_k\}$. Then, for all $n \geq k$.

$I_k \leq p_n \leq S_k$.

Now, a is the least upper bound of I_k ($a = \sup\{I_k\}$) and p_n is an upper bound for I_k $\Rightarrow a = \sup\{I_k\} \wedge p_n \geq I_k \Rightarrow a \leq p_n$. Likewise, b is the greatest lower bound of S_k ($b = \inf\{S_k\}$). and p_n is a lower bound for S_k $\Rightarrow b = \inf\{I_k\} \wedge p_n \leq S_k \Rightarrow b \geq p_n$. Putting this together,

$a \leq p_n \leq b$, $\forall n \Rightarrow a \leq b \Leftrightarrow$

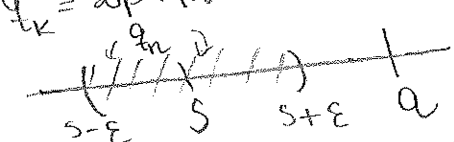
$$\boxed{\lim_n \inf\{p_n\} \leq \lim_n \sup\{p_n\}}$$ //

Proposition: As usual $\{p_n\} \subset \mathbb{R}$, $\{p_n\}$ bounded

(a) If $a > \lim_n \sup\{p_n\}$, then, $\exists\, k$ s.t. $\forall n \geq k$: $p_n < a$.

(b) If $a < \lim_n \sup\{p_n\}$, then, $\forall k : \exists n \geq k$ s.t.: $p_n > a$.

Pf: (a) Suppose $a > \lim_n \sup\{p_n\} = s$. By definition, If $q_k = \sup\{p_n : n \geq k\}$ then $q_k \xrightarrow[n\to\infty]{} s$. Hence, given $\varepsilon > 0 : \exists N : \forall n \geq N : |q_n - s| < \varepsilon$.



choose $\varepsilon = \frac{a-s}{2} > 0$. (since $a > s$ by hypothesis). Then, $\exists k : \forall n \geq k : |q_n - s| < \varepsilon$.

But by definition $q_k = \sup\{p_n : n \geq k\} \Rightarrow p_n \leq q_k$. Moreover $q_k < a$

$\Rightarrow p_n \leq q_k < a \Rightarrow p_n < a$. //

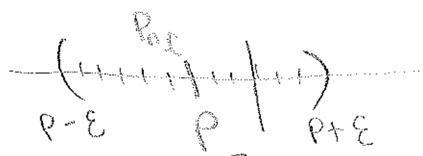
(b) Let $a < \lim_n \sup \{p_n\} = s$. $q_k = \sup\{p_n : n \geq k\}$. $q_k \xrightarrow[k \to \infty]{} s$.

Let $\varepsilon > 0$. $\exists N : \forall n \geq N : |q_n - s| < \varepsilon$.

Ask for this proof

Property: Let $\{p_n\}$ be a bounded sequence in $\mathbb{R}$ and $p \in \mathbb{R}$. then $\exists$ a subsequence $\{p_{n_e}\}$ of $\{p_n\}$ that converges to p if and only if $\forall \varepsilon > 0, \forall k \in \mathbb{N} : \exists n \geq k$ s.t. $|p_n - p| < \varepsilon$.

Pf: ($\Rightarrow$) Suppose $\{p_n\}$ has a subsequence $\{p_{n_e}\}$ s.t. $\{p_{n_e}\} \xrightarrow[n_e \to \infty]{} p$.



let $\varepsilon > 0$. Fix k. $\exists n_e \geq N$ arbitrary
$|p_{n_k} - p| < \varepsilon$.

$\exists p_{n_e}$, for n_e large enough. ; let $n_e = n$

$\Rightarrow |p_n - p| < \varepsilon$.

($\Leftarrow$) Suppose $\forall \varepsilon > 0 \, \forall k \in \mathbb{N} : \exists n \geq k$ s.t. $|p_n - p| < \varepsilon$. We wait to construct a subsequence $\{p_{n_e}\}$ that coverges to p.

First step: Let $\varepsilon = 1$, $k = 1$. Pick $n_1 \geq k$ s.t. $|p_{n_1} - p| < 1$.

SECOND step: Let $\varepsilon = \frac{1}{2}$, $k = n_1$. Pick $n_2 \geq n_1 = k$ s.t. $|p_{n_2} - p| < \frac{1}{2}$

$\vdots$

i th-step: let $\varepsilon = \frac{1}{i}$, $k = n_{i-1}$. Pick $n_i \geq n_{i-1} = k$ s.t. $|p_{n_i} - p| < \frac{1}{n}$.

$\vdots$

So the sequence $\{p_{n_i}\}$ $i = 1, 2, \ldots$, so constructed is s.t. it is a subsequence of $\{p_n\}$; and $\{p_{n_i}\} \xrightarrow[\text{as } n_i \to \infty]{} p$

theorem: As usual $\{p_n\} \subset \mathbb{R}$, $\{p_n\}$ bounded. then

(a) $\exists$ a subsequence $\{p_{n_e}\}$ of $\{p_n\}$ s.t. $p_{n_e} \to s = \lim_n \sup p_n$

(a') $\exists$ a subsequence $\{p_{n_j}\}$ of $\{p_n\}$ s.t. $p_{n_j} \to l = \lim_n \inf p_n$

(b) $\forall t \in \mathbb{R}$, if $\{p_{n_k}\}$ converges to t, then

$$\lim_n \inf p_n \leq t \leq \lim_n \sup p_n$$

(lim sup is the biggest convergent value for a subseq. and lim inf is the smallest

Analysis I. Enrique Areyan - Fall 2013

Pf: First note that ⓐ $\Rightarrow$ (a')

Suppose $\exists$ a subsequence $\{p_{n_\ell}\}$ of $\{p_n\}$ s.t. $p_{n_\ell} \to s = \limsup_n p_n$.

then, By definition, Given $\varepsilon > 0$, $\exists N: \forall n_\ell \geq N: |p_{n_\ell} - s| < \varepsilon$.

the subsequence we want to construct is $\{p_{n_j}\} = \{-p_{n_\ell}\}$. then $p_{n_j} \to l$, since:

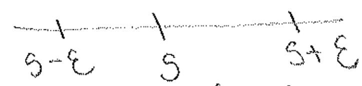
Let $\varepsilon > 0$. choose N s.t. $|p_{n_\ell} - s| < \varepsilon$ whenever $n_\ell \geq N$. then:

$$|p_{n_j} - l| = |-p_{n_\ell} + s| \quad \text{by definition of } \{p_{n_j}\} \text{ and } \underbrace{\liminf_n p_n}_{l} = \underbrace{-\limsup_n -p_n}_{-s}$$

$$= |s - p_{n_\ell}| < \varepsilon \Rightarrow \{p_{n_j}\} \underset{n_j \to \infty}{\to} l.$$

ⓐ the goal is to construct a subsequence $\{p_{n_\ell}\}$ of $\{p_n\}$ that converges to $\underset{\limsup_n p_n}{s}$.

Let $s = \limsup_n p_n$. Given $\varepsilon > 0$:



By previous proposition, since $s + \varepsilon > s = \limsup \{p_n\}$ then $\exists K$ s.t. $\forall n_\ell \geq K: p_{n_\ell} < s + \varepsilon$.

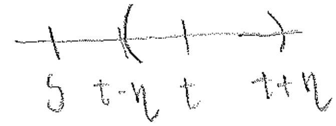
Pick such a K. then, $\forall n_\ell \geq K$:

; but p_{n_ℓ} could be smaller than $s-\varepsilon$ or s.

However, if that's the case: use the second part of previous propoposition; with some K as found before to pick $n_\ell \geq K$ s.t.:

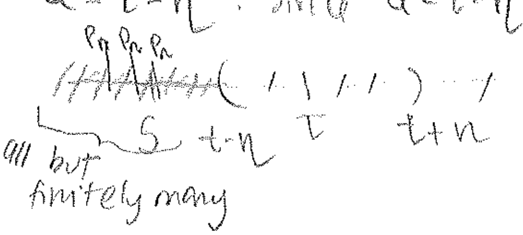
$p_{n_\ell} > s - \varepsilon$. By combining these two properties we get that. Given $\varepsilon > 0$. $\exists K$ s.t. $\forall n \geq K: |p_{n_\ell} - s| < \varepsilon \Leftrightarrow p_{n_\ell} \to s$.

ⓑ Let $t \in \mathbb{R}$. Suppose that $\{p_{n_k}\}$ converges to t. Let $s = \limsup_n p_n$. Suppose, for a contradiction that $t > s$. Let $\eta = \frac{t-s}{2} > 0$.



Apply previous proposition considering:

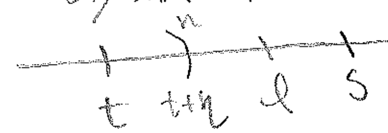
$a = t - \eta$. Since $a = t - \eta > s = \limsup p_n \Rightarrow \exists K: \forall n \geq K: \quad p_n < a = t - \eta$.



Pick such a K. then all but finitely many p_n are less that $t - \eta$, contradicting the fact that t is a limit point.

Hence, $t \leq s \Leftrightarrow t \leq \limsup_n p_n$

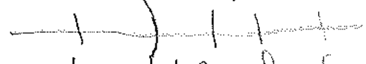
the same strategy works for showing that $t \geq \liminf_n p_n = l$. Suppose, for a contradiction that $t < l$. let $\eta = \frac{l - t}{2} > 0$



we proved $s \geq l$.

apply previous proposition considering:

$a = t + \eta$. Since $a = t + \eta < l < s \Rightarrow a < s \Rightarrow \forall K: \exists n \geq K$ s.t.: $\quad p_n > a = t + \eta$.



But by previous theoem, since $\{p_{n_k}\} \to t$ so $\forall \varepsilon > 0$ $\forall K \in \mathbb{N}$ $\exists n \geq K$ s.t. $p_n - t < \varepsilon$; But we have found $\forall K$: an $n \geq K$ s.t. ... a contradiction.

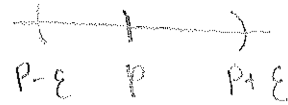
<u>Corollary</u>: $\{p_n\}$ bounded and $p_n \to p$ iff $p = \limsup_n p_n = \liminf_n p_n$

<u>Pf</u>: ($\Rightarrow$) Let $\{p_n\}$ be bounded and $p_n \to p$. Then, any subsequence of $\{p_n\}$ must converge to p. We proved that there is a subsequence converging to $\limsup_n p_n$ and a subsequence converging to $\liminf_n p_n$. Therefore,

$$\left.\begin{array}{l} \{p_{n_k}\} \to \limsup_n p_n \text{ and } \{p_{n_k}\} \to p \;\Rightarrow\; p = \limsup_n p_n \\ \{p_{n_j}\} \to \liminf_n p_n \text{ and } \{p_{n_j}\} \to p \;\Rightarrow\; p = \liminf_n p_n \end{array}\right\} \Rightarrow \boxed{p = \limsup_n p_n = \liminf_n p_n}$$

($\Leftarrow$) Suppose that $p = \limsup_n p_n = \liminf_n p_n$. Want to show $p_n \overset{?}{\to} p$

Let $\varepsilon > 0$.



. Since $p+\varepsilon > p$, all but finitely many $p_n < p+\varepsilon$.
Since $p-\varepsilon < p$, all but finitely many $p_n > p+\varepsilon$.

Hence, $\exists K$ s.t. $\forall n \geq K$ $|p_n - p| < \varepsilon$.

<u>THEOREM</u>: In $\mathbb{R}$, every Cauchy sequence converges.

<u>Pf</u>: Let $\{p_n\} \subset \mathbb{R}$ be a Cauchy sequence. We proved that all Cauchy sequences are bounded. Therefore, $\{p_n\}$ is bounded.
Moreover, by previous corollary, $\{p_n\}$ converges iff $\liminf_n p_n = \limsup_n p_n = p$.
We proved that $\limsup_n p_n \geq \liminf_n p_n$. So, we need only to show that $\liminf_n p_n \geq \limsup_n p_n$ to obtain the result. For this, let us prove, for an arbitrary $\varepsilon > 0$: $\limsup_n p_n \leq \liminf_n p_n + \varepsilon$.

SEQUENCES OF REAL NUMBERS: $\{X_n\}$.

Two essential theorems to keep in mind.

(I) Squeezing Principle: Given three sequences of real numbers: x_n, y_n and z_n. If $y_n \to L$ and $z_n \to L$ and $y_n \leq x_n \leq z_n$ then $x_n \to L$.

(II) If a sequence $\{x_n\}$ of real numbers is monotonic then $\{x_n\}$ converges iff it is bounded.

[monotonically increasing: $x_n \leq x_{n+1}$ $(n=1,2,3,\dots)$ monotonically decreasing: $x_n \geq x_{n+1}$ $(n=1,2,3,\dots)$]

Pf. (I). Let x_n, y_n, z_n be sequences of real numbers such that $y_n \to L$ and $z_n \to L$ and $y_n \leq x_n \leq z_n$. We want to show that $x_n \to L$. i.e., Given $\varepsilon > 0$ $\exists N: \forall n \geq N: |x_n - L| < \varepsilon$. We have that $y_n \to L$ and $z_n \to L$.

Hence, Given $\varepsilon > 0$: $\begin{cases} \exists N_1: \forall n \geq N_1: |y_n - L| < \varepsilon. \\ \exists N_2: \forall n \geq N_2: |z_n - L| < \varepsilon. \end{cases}$

Let $\varepsilon > 0$. Pick $n \geq \max(N_1, N_2)$. then: $|y_n - L| < \varepsilon \Rightarrow -\varepsilon < y_n - L < \varepsilon$
$|z_n - L| < \varepsilon \Rightarrow -\varepsilon < z_n - L < \varepsilon$

$L - \varepsilon < y_n \leq x_n \Rightarrow L - \varepsilon < x_n \Rightarrow -\varepsilon < x_n - L$
$x_n \leq z_n < L + \varepsilon \Rightarrow x_n < L + \varepsilon \Rightarrow x_n - L < \varepsilon$
$\Rightarrow -\varepsilon < x_n - L < \varepsilon \Rightarrow |x_n - L| < \varepsilon$

therefore $x_n \to L$.

(II) Let $\{x_n\}$ be a monotonic sequence of real numbers. [monotonically increasing. For the other case it is analogous]

($\Rightarrow$) by theorem 3.2(c); assuming $\{x_n\}$ converges then it is bounded.

($\Leftarrow$) Suppose $\{x_n\}$ is bounded. Let $S = \sup\{x_n\}$. We want to show that $x_n \to S$. (note here that $\{x_n\}$ is bounded so it is bounded above and is not empty so $\sup\{x_n\}$ exists. Moreover, for the case of monotonically decreasing we would use $\inf\{x_n\}$)

We want to show that, for $\varepsilon > 0$ $\exists N: \forall n \geq N: |x_n - S| < \varepsilon \iff -\varepsilon < x_n - S \wedge x_n - S < \varepsilon$.

Clearly, since S is the sup: $S \geq x_n$ $\forall n$. $\Rightarrow S + \varepsilon > x_n \Rightarrow x_n - S < \varepsilon$

It remains to show that given $\varepsilon > 0$: $\exists N: \forall n \geq N$: $-\varepsilon < x_n - S \iff S - \varepsilon < x_n$

Suppose for a contradiction that $\exists \varepsilon > 0: \forall N: \exists n \geq N$: $S - \varepsilon \geq x_n$: In particular $S - \varepsilon \geq x_N$; $\forall N$. Hence, $S - \varepsilon$ is an upper bound for $\{x_n\}$, but $S - \varepsilon < S$; where S is the least upper bound. therefore: Given $\varepsilon > 0$: $\exists N: \forall n \geq N$. $S - \varepsilon < x_n \iff -\varepsilon < x_n - S$ which together with $x_n - S < \varepsilon$ imply that $-\varepsilon < x_n - S < \varepsilon \iff |x_n - S| < \varepsilon$.

therefore x_n converges (to S).

THEOREM: (3.20)

(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$ ((c) $\Rightarrow$ (b)).

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ (Can be proved by l'Hopitila).

(e) If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$. (follows from (d)). take $\alpha = 0$.

Pf: (a) let $p > 0$. let $\varepsilon > 0$. Choose N so that $N > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$. Then, if $n > N$.

$$\frac{1}{n^p} < \frac{1}{N^p} < \varepsilon \Rightarrow \frac{1}{n^p} \to 0.$$

(b) claim: $\forall t > 0$ $(1+t)^n > 1+nt$, $\forall n$. (follows from Binomial theorem, or we can prove it).

Pf: Suppose $\varphi(t) = (1+t)^n - (1+C_n t)$.

We want to find C_n s.t. $\varphi(t) > 0$ for all t. Using Calculus:

$$\begin{cases} \varphi(0) = 0 \\ \varphi'(t) > 0 \Rightarrow \varphi(t) > \varphi(0) \quad \forall t > 0 \Rightarrow \varphi(t) > 0 \end{cases}$$ So then,

$\varphi'(t) = n(1+t)^{n-1} - C_n$; Choose $C_n = n \Rightarrow \varphi'(t) = n(1+t)^{n-1} - n$ (what if $n=1$?

$\varphi'(t) = n[(1+t)^{n-1} - 1] > 0.$

Hence, with the choice $C_n = n$, we prove the claim. □

Now, let $x_n = \sqrt[n]{x} - 1 \Rightarrow (x_n+1)^n = x \geq 1 + n x_n \Rightarrow x \geq 1 + n x_n$

three cases for x:

(I) $x > 1 \Rightarrow 0 \leq x_n \leq \frac{x-1}{n} \xrightarrow[n\to\infty]{} 0$

(II) $x = 1 \Rightarrow 1 \geq 1 + n x_n \Rightarrow 0 \geq x_n \Rightarrow 0 \leq x_n \leq 0 \Rightarrow x_n = 0$, $\forall n$.

(III) $x < 1$ (is this even defined? maybe for odd n's) $\Rightarrow \frac{1}{x} > 1 \Rightarrow \sqrt[n]{\frac{1}{x}} = \frac{1}{\sqrt[n]{x}} \to 1$ by first case

(c) Using same ideas as before:

$\varphi(t) = (1+t)^n - C_n t^2$; find C_n s.t $\varphi'(t) > 0$.

$\varphi'(t) = n(1+t)^{n-1} - 2C_n t \overset{\text{binomial theorem.}}{>} n(1+(n-1)t) - 2C_n t = n + n(n-1)t - 2C_n t$;

Let $C_n = \frac{n(n-1)}{2}$; then $\varphi'(t) = n + n(n-1)t - n(n-1)t = n > 0$. since $n \in \mathbb{N}$.

Using the value $c_n = \frac{n(n-1)}{2}$, we get:

$\varphi(t) = (1+t)^n - \frac{n(n-1)t^2}{2} > 1 = \varphi(0)$ (using calculus). $\forall t$.

$\Rightarrow (1+t)^n > 1 + \frac{n(n-1)t^2}{2}$. Now apply this to get:

$n = (1+x_n)^n > 1 + \frac{n(n-1)x_n^2}{2} \quad \forall n \Rightarrow n > 1 + \frac{n(n-1)x_n^2}{2}$

$\Rightarrow 2(n-1) > n(n-1)x_n^2 \Rightarrow \frac{2}{n} > x_n^2 \Rightarrow \sqrt{\frac{2}{n}} > x_n$

$0 \le x_n \le \left(\frac{2}{n}\right)^{1/2} \xrightarrow[\text{as } n\to\infty]{} 0$ (by (a)).

(d) Let $p > 0$ and $\alpha \in \mathbb{R}$. Pick k s.t. $k > \alpha$ and $k > 0$. Then, for $n > 2k$:

$(1+p)^n > \binom{n}{k}p^k = \frac{n(n-1)\cdots(n-k+1)}{k!}p^k > \frac{n^k p^k}{2^k k!}$ ← why?

binomial theorem.

$\Rightarrow (1+p)^n > \frac{n^k p^k}{2^k k!}$

Hence, $0 < \frac{1}{(1+p)^n} < \frac{2^k k!}{n^k p^k} \Rightarrow 0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k}$; since $\alpha - k < 0$, $\xrightarrow[n\to\infty]{} 0$ by (a).

(e) Take $\alpha = 0$. by (d).

$\lim_{n\to\infty} \frac{n^0}{(1+p)^n} = 0 \iff \lim_{n\to\infty}\left(\frac{1}{1+p}\right)^n = 0$; let $x = \frac{1}{1+p}$; then,

$\lim_{n\to\infty} x^n = 0$, when $|x| < 1$, since $p > 0$.

<u>THEOREM</u>: About number e.

Let $C_n = \left(1+\frac{1}{n}\right)^n$ then C_n is increasing and bounded above. In fact $\lim_n \left(1+\frac{1}{n}\right)^n = e$.

Since C_n is increasing and bounded, it converges.

<u>Pf</u>: (I) C_n is increasing. To prove this, let us use the fact.

If $0 < y < x$, then $x^r - y^r = (x-y)(x^{r-1} + x^{r-2}y + \cdots + xy^{r-2} + y^{r-1})$; replace y by x in RHS.

$< (x-y)(x^{r-1} + x^{r-2}x + \cdots + x x^{r-2} + x^{r-1})$

$= (x-y)(x^{r-1} + x^{r-1} + \cdots + x^{r-1} + x^{r-1})$

$= (x-y) r x^{r-1}$

$\Rightarrow x^r - y^r < (x-y) r x^{r-1} \Rightarrow \boxed{x^r < (x-y) r x^{r-1} + y}$

Note that the error is very big if x is far from y. Otherwise, the error is very small

Apply this inequality to:

$$x = 1+\frac{1}{n} \ ; \ y = 1+\frac{1}{n+1} \ ; \quad r = n+1$$

$$\Rightarrow x-y = 1+\frac{1}{n}-1-\frac{1}{n+1} = \frac{n(n+1)+(n+1)-n(n+1)-n}{n(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$$

then,

$$\left(1+\frac{1}{n}\right)^{n+1} < \left(1+\frac{1}{n+1}\right)^{n+1} + \left(\frac{1}{n(n+1)}\right)(n+1)\left(1+\frac{1}{n}\right)^{n}$$

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)C_n < C_{n+1} + \frac{1}{n}C_n \quad \Rightarrow$$

$$C_n\left(1+\frac{1}{n}-\frac{1}{n}\right) < C_{n+1} \quad \Rightarrow \quad \boxed{C_n < C_{n+1}}$$

$\to$ SEQUENCE IS increasing.

(II) C_n is bounded:

Use previous inequality with $r=n \Rightarrow x^n < y^n + (x-y)nx^{n-1}$, $0<y<x$, $\forall n$.

Let $x = 1+\frac{1}{2n}$ and $y = 1$. then,

since $n \geq 1$.

$$\left(1+\frac{1}{2n}\right)^n < 1 + \left(\frac{1}{2n}\right)n\left(1+\frac{1}{2n}\right)^{n-1} \leq 1+\frac{1}{2}\left(1+\frac{1}{2n}\right)^n$$

$$\Rightarrow \frac{1}{2}\left(1+\frac{1}{2n}\right)^n < 1 \Rightarrow \left(1+\frac{1}{2n}\right)^n < 2 \Rightarrow \left(1+\frac{1}{2n}\right)^{2n} < 4.$$

Hence, we have found an upper bound for odd terms of $\{(1+\frac{1}{n})^n\}$. But the sequence is increasing, so we have found a bound for C_n i.e., $C_n < 4$.

(I) & (II) $\Rightarrow$ C_n converges. $\exists \lim_n C_n = \lim_n \left(1+\frac{1}{n}\right)^n = e \leq 4$

<u>SERIES</u>: Given a sequence $\{a_k\}$, define $S_n = \sum_{k=1}^{n} a_k$. (if we write no limits, e.g., $\sum a_k = \sum_{k=1}^{\infty} a_k$, beginning at one). then

$$S = \lim_{n\to\infty} S_n \text{ (whenever it exists)} = \sum_{k=1}^{\infty} a_k.$$

<u>Observation</u>: the Cauchy criterion for convergence of sequences can be restated in the following form: Recall: Cauchy $\Leftrightarrow$ converges (in $\mathbb{R}$).

Cauchy: $\varepsilon>0 : \exists N : \forall n,m \geq N : |x_n - x_m| < \varepsilon$. For series:

$\varepsilon > 0 : \exists N : \forall n,m \geq N \ (m \geq n) : |S_m - S_n| < \varepsilon. \Leftrightarrow \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k\right| < \varepsilon$

$$\Leftrightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$$

therefore, $\sum a_n$ converges if and only if: $\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$.

Given $\varepsilon > 0$ $\exists N : \forall n,m \geq N \ (m \geq n)$:

In particular, if $n=m$ then we get $|a_n| < \varepsilon \quad (n \geq N)$.

which means:

THEOREM: if $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Pf: follows from Cauchy criterion by setting $n=m$.

Note that the other direction does not work, i.e., consider the

HARMONIC SERIES $\sum \frac{1}{n}$. Note that $\lim_{n\to\infty} \frac{1}{n} = 0$ (clearly), but $\sum \frac{1}{n}$ diverges

since: $1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq 2\cdot\frac{1}{4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq 4\frac{1}{8}} + \cdots$

$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = 1+1+1+\cdots \to \infty$

Hence, the series diverges since the partial sums can be made as big as you want.

theorem: Comparison test.

(a) If $|a_n| \leq c_n$, for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges then $\sum a_n$ diverges

((b) applies only to series of nonnegative terms a_n).

SERIES OF POSITIVE TERMS:

THEOREM: A series of positive terms (non-negative terms). ($\{S_n\} \Rightarrow$ partial sums).

converges $(\sum a_k = S)$ $\iff$ $\{S_n\}$ is bounded above

GEOMETRIC SERIES: If $|x|<1$ $\sum x^n = \frac{1}{1-x}$. o/w $|x| \geq 1$ $\sum x^n$ diverges. If $x \neq 1$:

Proof: Clearly, if $x=1$ then $\sum x^n = 1+1+1+\cdots \to \infty$.

$S_n = \sum_{k=0}^{n-1} x^k$; $xS_n = x\sum_{k=0}^{n-1} x^k = \sum_{k=0}^{n-1} x^{k+1}$; change variables: $m=k+1$. $k=0 \Rightarrow m=1$. $k=n-1 \Rightarrow m=n$.

$xS_n = \sum_{m=1}^{n} x^m$; relabel $m=k \Rightarrow xS_n = \sum_{k=1}^{n} x^k$.

then: $S_n - xS_n = \sum_{k=0}^{n-1} x^k - \sum_{k=1}^{n} x^k \Rightarrow S_n(1-x) = 1 - x^n \Rightarrow$ $\boxed{S_n = \frac{1-x^n}{1-x}}$

Two cases: (I) $0 \leq x < 1$; then $S_n \to \frac{1}{1-x}$ as $n \to \infty$.

(II) $x \geq 1$; then S_n is increasing so $S_n \to \infty$, as $n\to\infty$.

GEOMETRIC SERIES converges iff $|x| < 1$ $(-1 < x < 1)$.

Cauchy Criterion of convergence: let $a_k \geq 0$ be a decreasing sequence $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. $a_k \downarrow 0$. then,

$$\sum a_n \text{ converges} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges.}$$

Pf: ($\Leftarrow$) Suppose $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. By previous theorem, a series of nonnegative terms converges iff its partial sums are bounded. So we want to bound s_n knowing that t_k is bounded where:

$$s_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + a_n$$
$$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}.$$

Now, for $n < 2^k$. since a_k is decreasing.

$$s_n \leq a_1 + \underbrace{(a_2 + a_3)}_{<2a_2} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{<4a_4} + \cdots + \underbrace{(a_{2^k} + \cdots + a_{2^{k+1}-1})}_{<2^k a_{2^k}}$$
$$= a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k$$

$\Rightarrow s_n \leq t_k$; but t_k is bounded, so s_n is also bounded $\Rightarrow \sum a_k$ converges.

($\Rightarrow$) if $n > 2^k$.

$$s_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})$$
$$\geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})$$
$$\geq \frac{1}{2}a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} t_k$$

$\Rightarrow 2s_n \geq t_k$; but s_n is bounded so t_k is also bounded $\Rightarrow \sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Application of Cauchy criterion:

If $p > 0$, $\sum \frac{1}{n^p}$ converges if $p > 1$ diverges if $p \leq 1$.

Pf: If $p \leq 0$, then $\lim_n \frac{1}{n^p} \to \infty$ as $n \to \infty \Rightarrow \sum \frac{1}{n^p}$ diverges.

If $p > 0$ then $\frac{1}{n^p}$ is decreasing, so we can apply Cauchy criterion to $\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} \left(\frac{2}{2^p}\right)^k = \sum_{k=0}^{\infty} (2^{1-p})^k = \sum_{k=0}^{\infty} 2^{k(1-p)} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^k$

this is a geometric series, so it converges when $1-p < 0 \iff p > 1$. It diverges otherwise.

NOTE: SHORTER PROOF FOR Cauchy CRITERION:

$$\sum a_{2^k+1} + \cdots + a_{2^{k+1}} \leq \sum 2^k a_{2^k} \leq 2 \sum (a_{2^{k-1}} + \cdots + a_{2^k})$$

More applications: We have that:

$\sum \frac{1}{n}$ $\downarrow$ diverges

$\sum \frac{1}{n^p}$ $\downarrow$ converges if $p > 1$.

what about something in between? e.g. $\sum \frac{1}{n\log(n)}$ → goes slower to infinity than $\frac{1}{n}$ But still diverges.

Claim: $\sum \frac{1}{n\log(n)}$ diverges.

Pf: By Cauchy criterion: $\sum_{k=0}^{\infty} 2^k \frac{1}{2^k \log(2^k)} = \sum_{k=0}^{\infty} \frac{1}{\log(2^k)} = \sum_{k=0}^{\infty} \frac{1}{k\log(2)} = \frac{1}{\log(2)} \sum_{k=0}^{\infty} \frac{1}{k}$;

this is a harmonic series, so it diverges. □

REPEAT above reasoning:

$\sum \frac{1}{n\log(n)}$ $\downarrow$ diverges

$\sum \frac{1}{n\log(n)^p}$ $\downarrow$ converges if $p > 1$.

note that $\frac{1}{n\log(n)^p}$ goes slower to infinity than $\frac{1}{n\log(n)}$.

Claim: $\sum \frac{1}{n\log(n)^p}$ converges if $p > 1$.

Pf By Cauchy criterion: $\sum_{k=0}^{\infty} 2^k \frac{1}{2^k \log(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{(k\log(2))^p} = \frac{1}{\log(2)^p} \sum_{k=0}^{\infty} \frac{1}{k^p}$ □

this process can be continued: $\sum \frac{1}{n\log n \log(\log(n))}$ diverges whereas $\sum \frac{1}{n\log n \log(\log n)^2}$ converges. Therefore, there is no boundary (for convergence and divergence of series); you can always make an slower series (smaller terms) out of a divergent series that also diverges.

theorem: $\sum_{k=0}^{\infty} \frac{1}{k!} = e$. Pf: See Rudin. Uses Binomial theorem

the above theorem has two important uses:

(I) Allows for efficient computation of e.

(II) e is irrational (trascendental).

(I) $S_n = \sum_{k=0}^{n} \frac{1}{k!}$. $\Rightarrow e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$

$< \frac{1}{(n+1)!} \underbrace{\left\{ 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \right\}}_{\text{geometric series converging to } \frac{1}{1-\frac{1}{n+1}} = \frac{1}{\frac{n}{n+1}} = \frac{n+1}{n}} = \frac{1}{n!\,n}$

$\Rightarrow 0 < e - S_n < \frac{1}{n!\,n}$ → measures the error of approximation,

e.g., $0.\underbrace{0000}5$ → correct on all of these

(II) Suppose for a contradiction that e is rational. $e = p/q$, p, q integers (positive)

we had that $0 < e - S_n < \frac{1}{n!n}$; this is true for any integer n, in particular for $n = q$:

$0 < e - S_q < \frac{1}{q!q} \Rightarrow 0 < (e - S_q)q!q < 1$. Analyze the number:

$(e - S_q)q!q$. this is going to be an integer if $(e - S_q)q!$ is an integer

Then, $eq! = \frac{p}{q} \cdot q! = p(q-1)! \Rightarrow$ an integer.

$$S_q q! = \left(\sum_{k=1}^{q} k!\right) q! = \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{q!}\right) \cdot q!$$

$$= q! + q! + q(q-1)\cdots(q \cdot (q-3)) + \dots + 1 \Rightarrow \text{an integer.}$$

So $(e - S_q)q!$ is an integer $\Rightarrow (e - S_q)q! \cdot q$ is an integer, so we have found an integer n s.t $0 < n < 1$, a contradiction $\Rightarrow e$ is irrational.

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the root and ratio tests:

THE ROOT TEST: Given $\sum_n a_n$, let $\alpha = \limsup_n \sqrt[n]{|a_n|}$. then:

If $\begin{cases} \alpha < 1, & \sum_n a_n \text{ converges} \\ \alpha > 1, & \sum_n a_n \text{ diverges} \\ \alpha = 1, & \text{the test provides no information.} \end{cases}$

Pf: Let $\alpha = \limsup_n \sqrt[n]{|a_n|}$. CASE BY CASE:

If $\alpha < 1$. then: $\alpha \; \beta \; 1$. By properties of limsup, $\exists N : \forall_n \geq N$. when $n \geq N$. $\sqrt[n]{|a_n|} < \beta \Leftrightarrow |a_n| < \beta^n$. Now, since $0 < |a_n| < \beta^n \Rightarrow 0 < \beta^n \Rightarrow 0 < \beta$.

Hence, $0 < \beta < 1$, which means that $\sum \beta^n$ converges and by comparison test $\sum a_n$ converges.

If $\alpha > 1$ then: $1 \; \alpha$ infinitely many $\sqrt[n_k]{|a_{n_k}|}$. By properties of lim sup, there exists a convergent subsequence $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. therefore, for big enough n_k, $\sqrt[n_k]{|a_{n_k}|} > 1 \Rightarrow |a_{n_k}| > 1$ so the whole sequence does not go to zero, (nth term test).

$a_n \not\to 0$, which means that $\sum_n a_n$ diverges.

If $\alpha = 1$, Consider $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$ then, $\limsup \sqrt[n]{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n}} = 1$ and $\sum \frac{1}{n}$ diverges, but, $\limsup \sqrt[n]{\frac{1}{n^2}} = \limsup \frac{1}{\sqrt[n]{n^2}} = \limsup \frac{1}{\sqrt[n]{n}\sqrt[n]{n}} = 1$ and

$\sum \frac{1}{n^2}$ converges. So $\alpha = 1$ gives no information.

Analysis I: Enrique Areyan - Fall 2013

RATIO TEST: Given $\sum a_n$, let $\alpha = \limsup_n \left|\frac{a_{n+1}}{a_n}\right| = \limsup_n \frac{|a_{n+1}|}{|a_n|}$. Then:

$$\text{If } \begin{cases} \alpha < 1, & \sum a_n \text{ converges} \\ \alpha > 1, & \sum a_n \text{ diverges} \\ \alpha = 1, & \text{the test provides no information.} \end{cases}$$

Pf: Let $\alpha = \limsup_n \frac{|a_{n+1}|}{|a_n|}$. CASE by CASE:

If $\alpha < 1$ then $\alpha \quad \beta \quad 1$. By properties of limsup, $\exists N$ s.t. $\forall n \geq N$:

$0 < \frac{|a_{n+1}|}{|a_n|} < \beta$. Pick such an N and consider for $k \geq 1$:

$$\frac{|a_{N+k}|}{|a_N|} = \frac{|a_{N+k}|}{|a_{N+(k-1)}|} \frac{|a_{N+(k-1)}|}{|a_{N+(k-2)}|} \cdots \frac{|a_{N+1}|}{|a_N|}$$

$$< \beta \cdot \beta \cdots \beta = \beta^k$$

$\Rightarrow \frac{|a_{N+k}|}{|a_N|} < \beta^k \Rightarrow |a_{N+k}| < \beta^k |a_N|$. Now, since $0 < \beta < 1$, $\sum \beta^k |a_N| = |a_N| \sum \beta^k$ converges. therefore, $\sum |a_{N+k}|$ converges. this is the tail of the sum and therefore $\sum a_n$ converges.

If $\alpha > 1$ then $1 \quad \alpha$. By properties of limsup there exists a convergent subsequence $\{a_{n_k}\}$ s.t. $\left|\frac{a_{n_k+1}}{a_{n_k}}\right| \to \alpha$. Hence, $\frac{|a_{n_k+1}|}{|a_{n_k}|} > 1$

$\Rightarrow |a_{n_k+1}| > |a_{n_k}|$: for n_k large enough. Clearly $a_n \not\to 0$, $\sum a_n$ diverges.

If $\alpha = 1$ consider, $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$. then $\limsup \frac{\frac{1}{n+1}}{\frac{1}{n}} \cdot \limsup \frac{n}{n+1}$

Since $\lim \frac{n}{n+1} = 1 \Rightarrow \limsup \frac{n}{n+1} = 1$ and we know $\sum a_n$ diverges.

Also, $\limsup \frac{\frac{1}{n+n^2}}{\frac{1}{n}} = \limsup \frac{n}{(n+1)^2}$, since $\lim \frac{n}{(n+1)^2} = \lim \frac{1}{2}$???

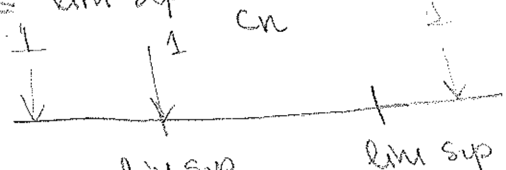
and we know $\longrightarrow \sum \frac{1}{n^2}$ converges.

THEOREM: For any sequence $\{c_n\}$ of positive numbers,

(a) $\liminf_n \frac{c_{n+1}}{c_n} \leq \liminf_n \sqrt[n]{c_n}$

(b) $\limsup_n \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$

(Pf in Rudin)



REMARKS: (1) Ratio test usually easier to apply than root test.
(2) Root test has wider scope.

Ratio test converges $\Rightarrow$ Root converges

Root Test inconclusive $\Rightarrow$ Ratio ??? test inconclusive

POWER SERIES:

Definition: Given a sequence $\{a_n\}$ of real numbers, the series $\sum_{n=0}^{\infty} a_n x^n$ is called a <u>power series</u>. The numbers a_n are called the coefficients of the series.

Convergence depends on x. (Radius of convergence).
We may think of a power series as a sophisticated geometric series with coefficients.

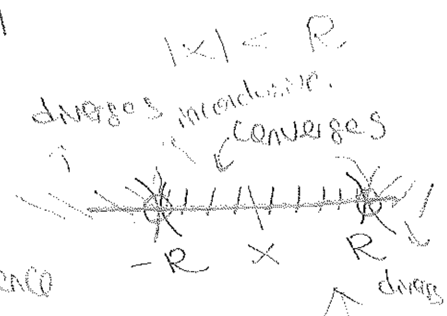
<u>Application</u> of Ratio & Root Tests to power series: $\sum_{n=0}^{\infty} a_n x^n$

<u>RATIO TEST to POWER SERIES</u>: Consider:

$$\limsup_n \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \limsup_n \frac{|a_{n+1}x|}{|a_n|} = |x| \limsup_n \frac{|a_{n+1}|}{|a_n|},$$

So, our power series <u>converges absolutely</u> provided that

$$|x| \limsup_n \frac{|a_{n+1}|}{|a_n|} < 1 \implies |x| < \frac{1}{\limsup_n \frac{|a_{n+1}|}{|a_n|}}$$



Let $R = \dfrac{1}{\limsup \frac{|a_{n+1}|}{|a_n|}}$, we call R the <u>radius</u> of convergence

<u>ROOT TEST to POWER SERIES</u>: Consider:

$$\limsup_n \sqrt[n]{|a_n x^n|} = \limsup_n \sqrt[n]{|a_n|}\sqrt[n]{|x^n|} = |x| \limsup_n \sqrt[n]{|a_n|},$$

So, our power series converges provided that

SAME picture AS

$$|x| \limsup_n \sqrt[n]{|a_n|} < 1 \implies |x| < \frac{1}{\limsup_n \sqrt[n]{|a_n|}}$$

Let $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$, we call R the <u>Radius</u> of convergence

<u>Note</u>: if $\limsup = 0 \implies R = +\infty$, if $\limsup = +\infty$ then $R = 0$.

<u>Examples</u> (I) $\sum n^n x^n$. $\limsup_n \sqrt[n]{|n^n|} = \limsup_n |n| = +\infty \implies R = 0$.

(I) $\sum \frac{x^n}{n!}$. $\limsup_n \frac{\left|\frac{1}{(n+1)!}\right|}{\left|\frac{1}{n!}\right|} = \limsup_n \frac{|n!|}{|(n+1)!|} = \limsup_n \frac{1}{|n|} = 0 \implies R = +\infty$.

(II) $\sum x^n$ $\limsup_n 1 = 1 \implies R = 1$.

(V) $\sum \frac{x^n}{n}$. $\limsup_n \frac{\frac{1}{n+1}}{\frac{1}{n}} = \limsup \frac{n}{n+1} = 1 \implies R = 1$

Summation by Parts (Abel) (Related to integration by parts).

Given two sequences $\{a_n\}, \{b_n\}$, $n=0,1,2,\ldots$ Let $A_n = \sum_{k=0}^{n} a_k$, $(A_{-1}=0)$

If $0 \le p \le q$: $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$.

Pf: $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n$; since $A_n - A_{n-1} = \sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k = a_n$.

$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$; distributing b_n

$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$; starting second sum from $p-1$ instead of p

$= \left(\sum_{n=p}^{q-1} A_n b_n\right) + A_q b_q - \left(\sum_{n=p}^{q-1} A_n b_{n+1}\right) - A_{p-1} b_p$ taking out first terms

$= \left(\sum_{n=p}^{q-1} A_n b_n - A_n b_{n+1}\right) + A_q b_q - A_{p-1} b_p$ grouping

$= \left(\underbrace{\sum_{n=p}^{q-1}}_{\text{integrated}} A_n \underbrace{(b_n - b_{n+1})}_{\text{differentiation}}\right) + \underbrace{A_q b_q - A_{p-1} b_p}_{\text{"integrated" terms}}$.

Applications. Suppose

(a) $\{A_n\}$ is bounded, i.e., $\exists M$ s.t. $|A_n| < M$, $\forall n$

(b) $b_n \downarrow 0$, b_n positive, $\lim b_n = 0$ then.

$\sum a_n b_n$ converges

Pf: We want to show that $\sum a_n b_n$ is Cauchy, i.e., $\forall \varepsilon > 0 : \exists N : \forall n,m \ge N$ $(m \ge n)$: $\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$

$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$ by previous theorem

$\le \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})\right| + |A_q b_q| + |A_{p-1} b_p|$ Triangular ineq.

$\le \sum_{n=p}^{q-1} |A_n| (b_n - b_{n+1}) + |A_q| b_q + |A_{p-1}| b_p$ since $b_i > 0$, descending

$\le M\left[\left(\sum_{n=p}^{q-1}(b_n - b_{n+1})\right) + (b_q + b_p)\right]$ since $\{A_n\}$ is bounded

$= M\left[(b_p - b_{p+1}) + (b_{p+1} - b_{p+2}) + \cdots + (b_{q-1} - b_q) + b_q + b_p)\right]$ expanding sum

$= M[b_p - \cancel{b_q} + b_p + \cancel{b_q}] \Rightarrow \left|\sum_{n=p}^{q} a_n b_n\right| \le 2 M b_p$

Given $\varepsilon > 0$, pick N s.t. $b_k < \frac{\varepsilon}{2M}$, $\forall k \geq N$. We can do this b/c $\lim b_n = 0$.
then, for $q \geq p \geq N$ $\left|\sum_{n=p}^{q} a_n b_n\right| \leq 2Mb_p \leq 2M \cdot \frac{\varepsilon}{2M} \leq \varepsilon$. Hence, $\sum a_n b_n$ converges.

Alternating SERIES: Suppose:

(a) $|c_1| \geq |c_2| \geq \cdots$

(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $(m = 1, 2, 3, \ldots)$ i.e., the terms alternate in sign.

(c) $\lim_{n\to\infty} c_n = 0$ then $\sum c_n$ converges.

Pf: Apply previous theorem with $a_n = (-1)^n$ and $b_n = |c_n|$
Note that $A_n = 1$ or 0 and thus $\{A_n\}$ is bounded.

Ex: $c_0 - c_1 + c_2 - c_3 + c_4 - \cdots \to$ converges (If previous hypothesis are met).
$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \to$ converges (to $\log(2)$).

ABSOLUTE CONVERGENCE:

Def: the series $\sum a_n$ is said to converge absolutely if the series $\sum |a_n|$ converges

THEOREM: If $\sum |a_n|$ converges then $\sum a_n$ converges
(absolute convergence $\Rightarrow$ converges)

Pf: follows from $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k|$, using Cauchy criteria.
Note that the converse is not true. e.g $\sum \frac{(-1)^n}{n}$ converges but $\sum \frac{1}{n}$ does not.

ADDITION AND MULTIPLICATION OF SERIES:

FACT: If $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$ and
$\sum c a_n = cA$, for a constant c.

Proof: let $\sum a_n = A$ and $\sum b_n = B$. then $\lim_{n\to\infty} A_n = A$ (k partial sum) and
$\lim_{n\to\infty} B_n = B$. Hence $\lim_n (A_n + B_n) = \lim_n A_n + \lim_n B_n = A + B = \sum (a_n + b_n)$
and $\lim_n cA_n = c \lim_n A_n = cA = c\sum A_n$. $\cancel{/}$.

Multiplication: Note that there is no unique way to define the product of two series. We will use the Cauchy Product.

Definition: Cauchy Product. Given $\sum a_n$ and $\sum b_n$, define

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \ldots)$$

call $\sum c_n$ the product of the two series.

Example: Consider $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Let us take the following product:

$e^{2x} = e^x e^x = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$. Using Cauchy product:

$$\left(\sum_{k=0}^{n} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{n} \frac{x^k}{k!}\right) = \left(\frac{x^0}{1} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right)\left(\frac{x^0}{1} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right)$$

$$= \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{x^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{x^n}{k!(n-k)!} = \sum_{k=0}^{n} \frac{x^n}{k!(n-k)!} \frac{n!}{n!}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{x^n}{n!} = \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} = \frac{x^n}{n!} 2^n = \frac{(2x)^n}{n!}$$

$\Rightarrow \sum \frac{x^n}{n!} \sum \frac{x^n}{n!} = \sum \frac{(2x)^n}{n!} = e^{2x}$, as expected.

The QUESTION is: Does the Cauchy Product behaves nicely?. Need hypothesis to make it work.

ANSWER: Not quite, i.e., we need to add hypothesis to make it work.

Consider $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$. Note that this is an alternating series with general term $\frac{1}{\sqrt{k}} \to 0$, so it converges.

However, the Cauchy product of $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$ with itself does not converge despite of the fact that each series converges.

Let us form the Cauchy product: Let $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}$

$$c_n = \sum_{k=0}^{n-1} a_n a_{n-k} = \sum_{k=0}^{n-1} \frac{(-1)^{k-1}}{\sqrt{k}} \cdot \frac{(-1)^{n-k-1}}{\sqrt{n-k}} = \sum_{k=0}^{n-1} \frac{(-1)^n}{\sqrt{k}\sqrt{n-k}}$$; but then

$$|c_n| = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}\sqrt{n-k}} \geq \frac{1}{\sqrt{n-1}\sqrt{n-1}}(n-1) = \frac{(n-1)}{(n-1)} = 1$$, Hence $c_n \not\to 0$; so the series does not converge.

LEMMA: Consider the sequences: $\{\alpha_n\}, \{\beta_n\}$ to be such that:

$\sum |\beta_n| < \infty$ and $\alpha_k \to 0$. Let us prove:

(a) $\gamma_n = \sum_{k=0}^{n} \alpha_k \beta_{n-k} \to 0$ (the Cauchy Product goes to zero).

(b) If $\alpha_n \to \alpha$ then $\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k \to \alpha$.

(c) If $\alpha_n \to \alpha$ and $\beta_n \to \beta$ then $\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k \beta_{n-k} \to \alpha\beta$.

Pf:

(a) Let $\beta = \sum_{n=0}^{\infty} |\beta_n|$. General Strategy: Look at
(I) Cancelling terms
(II) contributions.

Take Cauchy Product:

$$\sum_{k=0}^{n} |\alpha_k||\beta_{n-k}| = \underbrace{\sum_{k=0}^{n_0} |\alpha_k||\beta_{n-k}|}_{I(n_0)} + \underbrace{\sum_{k=n_0+1}^{n} |\alpha_k||\beta_{n-k}|}_{J(n_0)}$$

for $n_0 < n$
Breaking the sum into two pieces.

Given $\varepsilon > 0$, Pick n_0 (this is fixed) s.t. $|\alpha_k| \leq \eta\varepsilon$, $\forall k \geq n_0$. (We can do this b/c $\alpha_k \to 0$). (η is a parameter to allow for more "room" when computing).

Then,
on the one hand:

$$J(n_0) \leq \eta\varepsilon \sum_{k=n_0+1}^{n} |\beta_{n-k}| \leq \eta\varepsilon\beta \leq \frac{\varepsilon}{2}.$$

since $\beta = \sum |\beta_n|$
So choose $\eta\beta < \frac{1}{2}$

On the other: $I(n_0) \leq \alpha \sum_{k=0}^{n_0} |\beta_{n-k}|$, where $\alpha = \sup_k |\alpha_k| < \infty$ b/c $\alpha_k \to 0$ so α_k bounded.

$\leq \alpha \sum_{k=n-n_0}^{n} |\beta_k|$, change of variables

$\leq \alpha \sum_{k=n-n_0}^{\infty} |\beta_k|$, adding the tail of the sum.

But $\sum \beta_n$ converges, so pick n s.t. $\alpha \sum_{k=n-n_0}^{\infty} |\beta_k| < \frac{\varepsilon}{2}$. this n exists since the tail of a convergent series converges.

$\Rightarrow I(n_0) < \frac{\varepsilon}{2}$

$\Rightarrow I(n_0) + J(n_0) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Rightarrow \gamma_n \to 0$. (Cauchy Product converges)

(b) Let $\alpha_n \to \alpha$. Note that we may assume that $\alpha = 0$, since we can replace α_k by $(\alpha_k - \alpha)$, thus, if we have the result for zero, then

$$\frac{1}{n+1} \sum_{k=0}^{n} (\alpha_k - \alpha) = \frac{1}{n+1} \sum_{k=0}^{n} \alpha_k - \alpha = \frac{1}{n+1} \sum_{k=0}^{n} \alpha_k \to 0 \Rightarrow \frac{1}{n+1} \sum_{k=0}^{n} \alpha_k \to \alpha$$

So, let us prove that $\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k \to 0$. Strategy: compare large & small values,

$$\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k = \underbrace{\frac{1}{n+1} \sum_{k=0}^{n_0} \alpha_k}_{I(n_0)} + \underbrace{\frac{1}{n+1} \sum_{k=n_0+1}^{n} \alpha_k}_{J(n_0)}$$

$I(n_0)$ ~ small values.
Use that convergent seq → bounded
$J(n_0)$ ~ large values.

Given $\varepsilon > 0$, Pick n_0 (fixed) s.t.

$|\alpha_k| < \frac{\varepsilon}{2}$ $\forall k \geq n_0$, which exists because $\alpha_k \to 0$

$$|J(n_0)| = \left|\frac{1}{n+1}\sum_{k=n_0+1}^{n}\alpha_k\right| \leq \left|\frac{1}{n+1}\right|\sum_{k=n_0+1}^{n}|\alpha_k| = \frac{1}{n+1}\sum_{k=n_0+1}^{n}|\alpha_k| \text{ ; since } n+1>0.$$

$$\leq \frac{1}{n+1}\sum_{k=n_0+1}^{n}\frac{\varepsilon}{2} = \frac{1}{n+1}\frac{\varepsilon}{2}(n-n_0+1) < \frac{\varepsilon}{2} \quad \text{because } \frac{n-n_0-1}{n+1} < 1$$

$$|I(n_0)| = \left|\frac{1}{n+1}\sum_{k=0}^{n_0}\alpha_k\right| \leq \left|\frac{1}{n+1}\right|\sum_{k=0}^{n_0}|\alpha_k| = \frac{1}{n+1}\sum_{k=0}^{n_0}|\alpha_k|$$

Let $\alpha = \sup|\alpha_k|$. α exists because $\alpha_k \to 0$ so it is bounded.

$$\leq \frac{1}{n+1}\sum_{k=0}^{n_0}\alpha = \frac{1}{n+1}\alpha(n_0+1), \quad \underbrace{\text{Pick } n}_{???} \text{ s.t. } \frac{1}{n+1}\alpha(n_0+1) < \frac{\varepsilon}{2}$$

??? $\to$ can we pick this n???

$\Rightarrow$

$$\left|\frac{1}{n+1}\sum_{k=0}^{n+1}\alpha_k\right| = |I(n_0)+J(n_0)|$$
$$\leq |I(n_0)|+|J(n_0)| \qquad \Rightarrow \frac{1}{n+1}\sum_{k=0}^{n+1}\alpha_k \to 0.$$
$$\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

ⓒ Let $\alpha_n \to \alpha$, $\beta_n \to \beta$. I want to show that $\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k\beta_{n-k} \to \alpha\beta$

$$\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k\beta_{n-k} = \left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k\beta_{n-k}\right) - \left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha\beta_k\right) + \left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha\beta_k\right)$$

$$= \left(\frac{1}{n+1}\sum_{k=0}^{n}\alpha_k\beta_{n-k} - \frac{1}{n+1}\sum_{k=0}^{n}\alpha\beta_{n-k}\right) + \frac{1}{n+1}\sum_{k=0}^{n}\alpha\beta_k \quad \text{(change of variables)}$$

grouping Terms

$$= \underbrace{\frac{1}{n+1}\sum_{k=0}^{n}(\alpha_k-\alpha)\beta_{n-k}}_{I(n)} + \underbrace{\frac{1}{n+1}\sum_{k=0}^{n}\alpha\beta_k}_{J(n)}$$

$$J(n) = \frac{1}{n+1}\sum_{k=0}^{n}\alpha\beta_k = \frac{\alpha}{n+1}\sum_{k=0}^{n}\beta_k \to \alpha\beta \text{ , by part (b), since } \beta_n \to \beta$$

$$I(n) \leq \frac{1}{n+1}\sum_{k=0}^{n}|\alpha_k-\alpha||\beta_{n-k}| \text{ ;}$$
$$\leq \frac{1}{n+1}M\sum_{k=0}^{n}|\alpha_k-\alpha| \text{ , where } M \text{ is a bound for } \beta_n.$$
$$\leq 0 \qquad \text{since } \alpha_k \to \alpha.$$

$$\Rightarrow \frac{1}{n+1}\sum_{k=0}^{n}\alpha_k\beta_{n-k} = I(n)+J(n) \to 0+\alpha\beta = \alpha\beta.$$

So the product of these to sequences behaves nicely.

Mercer's theorem (1875): Suppose

ⓐ $A = \sum a_n$

ⓑ $B = \sum |b_n|$; $\sum |b_n|$ converges absolutely.

ⓒ $C_n = \sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0, 1, 2, \dots)$. Then

$\sum C_n = (\sum a_n)(\sum b_n) = AB$ (that is, the product of two convergent series converges - and to the right value - if at least one of the two series converges absolutely).

Pf: Let $A_n = \sum_{k=0}^{n} a_k$,

$B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$ $\left(c_k = \sum_{\ell=0}^{k} a_\ell b_{k-\ell}\right)$

Let $\alpha_n = A_n - A$, $\beta_n = b_n$. We need an expression of C_n.

$c_0 = a_0 b_0$

$c_1 = a_0 b_1 + a_1 b_0$

$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$

$\vdots$

$c_{n-2} = a_0 b_{n-2} + a_1 b_{n-3} + \cdots + a_{n-2} b_0$

$c_{n-1} = a_0 b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-1} b_0$

$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$

You can start from bottom corner or from the top.

$\left(A_0 b_n = a_0 b_n, \; A_1 b_{n-1} = (a_0 + a_1) b_{n-1} = a_0 b_{n-1} + a_1 b_{n-1}, \; A_2 b_{n-2} = (a_0 + a_1 + a_2) b_{n-2} = a_0 b_{n-2} + a_1 b_{n-2} + a_2 b_{n-2} \; \dots \right)$

$\sum c_n = A_0 b_n + A_1 b_{n-1} + A_2 b_{n-2} + A_3 b_{n-3} + \cdots + A_n b_0$ Dove tailing replacing $\alpha_n = A_n - A$

$= (\alpha_0 + A) b_n + (\alpha_1 + A) b_{n-1} + \cdots + (\alpha_n + A) b_0$

$= (\alpha_0 b_n + \alpha_1 b_{n-1} + \cdots + \alpha_n b_0) + A B_n$

Now, since $B = \sum |b_n|$ converges absolutely, $\sum_{k=0}^{n} \alpha_k b_{n-k} \to 0$ (part ⓐ)

$\sum C_n \to 0 + AB$, since $B_n \to B$; hence $\boxed{\sum C_n = AB}$

THEOREM. (Abel, 1826).

If: $\sum a_n \to A$, $\sum b_n \to B$, $\sum c_n \to C$ and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ then $C = AB$.

Pf: (Cesaro, 1870).

$C_k = A_0 b_k + A_1 b_{k-1} + \cdots + A_k b_0$, by previous proof.

$$\frac{C_0 + C_1 + \cdots + C_n}{n+1} = \frac{A_0 B_n + A_1 B_{n-1} + \cdots + A_n B_0}{n+1}$$

$\downarrow$ by part (b) $\qquad \downarrow$ by part (c)

$C \qquad = \qquad AB$

Applications:

(I) $\cos(\pi\sqrt{n^2+n})$ is this sequence convergent?

(II) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(x+n)}$ where does it converge? converge absolutely?

(I) $\cos(\pi\sqrt{n^2+n}) = \cos([\pi\sqrt{n^2+n} - \pi n] + \pi n)$

$= \cos(\pi\sqrt{n^2+n} - \pi n)\overset{\pm 1}{\cos(\pi n)} - \sin(\pi\sqrt{n^2+n})\overset{0}{\sin(\pi n)}$

$= \pm\cos(\pi\sqrt{n^2+n} - \pi n)$.

But,

$$\pi\sqrt{n^2+n} - \pi n = \pi\sqrt{n^2+n} - \pi n \cdot \frac{\pi\sqrt{n^2+n} + \pi n}{\pi\sqrt{n^2+n} + \pi n} = \frac{(\pi\sqrt{n^2+n})^2 - (\pi n)^2}{\pi\sqrt{n^2+n} + \pi n}$$

$$= \frac{\pi^2(n^2+n) - \pi^2 n^2}{\pi\sqrt{n^2+n} + \pi n} = \frac{\pi^2 n}{\pi\sqrt{n^2+n} + \pi n} = \frac{\pi^2}{\pi}\left(\frac{n}{\sqrt{n^2+n}+n}\right)$$

$$= \pi\left(\frac{n}{\sqrt{n^2+n}+n}\right) = \pi\left(\frac{(n)/n}{[\sqrt{n(n+1)}+n]/n}\right)$$

$$= \pi\left(\frac{1}{\sqrt{\frac{n(n+1)}{n^2}}+\frac{n}{n}}\right) = \pi\left(\frac{1}{\sqrt{\frac{n+1}{n}}+1}\right) = \pi\left(\frac{1}{\sqrt{\frac{n}{n}+\frac{1}{n}}+1}\right)$$

$$= \pi\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right), \text{ So } \lim_{n\to\infty}\left(\pi\sqrt{n^2+n} - \pi n\right) = \frac{\pi}{2}, \text{ which means}$$

$$\lim_{n\to\infty}\cos(\pi\sqrt{n^2+n}) = \lim_{n\to\infty} \pm\cos(\pi\sqrt{n^2+n} - \pi n) = \pm\cos\left(\frac{\pi}{2}\right) = \boxed{0}$$

(II) Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$.

At $x=0$ we get $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges by the Alternating Test.

In fact if $x>0$, we get an alternating series with general term $\frac{1}{x+n}$, $x>0$, $\to 0$ as $n\to\infty$, so the series converges (Alternating Test)

For all $x \neq -k$, where k is an integer, the series eventually alternates, so it also converges (eventually $x+n>0$, so $\frac{1}{x+n}\to 0$).

For absolutely: ???

CONTINUITY:

Definition: Consider (X, d_x) and (Y, d_y) to be metric spaces. $\rightarrow E = D(f)$
Suppose $f: E \subset X \rightarrow Y$. let $p \in X$, be a limit point of E.
We say $\lim_{x \to p} f(x) = q$, where $q \in Y$, If: ; Excludes the case when $x = p$
Given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $d_y(f(x), q) < \varepsilon$ provided that $0 < d_x(x, p) < \delta$. $\forall x \in E$.

Note that in this definition it not need to be the case that $p \in D(f)$. $(D(f) =$ domain of $f)$. Moreover, f does not need to be defined at $x = p$.

THEOREM (limit using sequences) let X, Y, E, f and p be as before.

$$\lim_{x \to p} f(x) = q \iff \forall \{p_n\} \subset D(f): \lim_{n \to \infty} f(p_n) = q$$
$$p_n \neq p,\ p_n \to p$$

Pf: $(\Rightarrow)$. Suppose $\lim_{x \to p} f(x) = q$. Choose $\{p_n\} \subset D(f)$ s.t. $p_n \neq p$, $p_n \to p$

So we have:
(I) Given $\varepsilon > 0$, there exists $\delta > 0$ s.t. If $0 < d_x(x, p) < \delta$ then $d_y(f(x), q) < \varepsilon$.
(II) Given $\varepsilon > 0$, there exists N s.t. for every $n \geq N$: $p_n \neq p$: $0 < d_x(p_n, p) < \varepsilon$.

We want to show: Given $\varepsilon > 0$, there exists $\delta > 0$ s.t. If $0 < d_x(x, p_n) < \delta$ then $d_y(f(p_n), q) < \varepsilon$.

BEGIN: Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $d_y(f(x), q) < \varepsilon$ whenever $0 < d_x(x, p) < \delta$
Now, for such a δ, choose N s.t. $0 < d_x(p_n, p) < \delta$ whenever $n \geq N$.
thus, for n large enough $(n \geq N)$. we have that
$0 < d_x(p_n, p) < \delta$ this implies that $d_y(f(p_n), q) < \varepsilon$.
which means that $\lim_{n \to \infty} f(p_n) = q$.

$(\Leftarrow)$ By contradiction, suppose is not the case that
$\forall \varepsilon > 0: \exists \delta > 0$: If: $0 < d_x(x, p) < \delta$ then $d_Y(f(x), q) < \varepsilon$.
then: $\exists \varepsilon > 0: \forall \delta > 0$: $0 < d_x(x, p) < \delta$ and $d_Y(f(x), q) \geq \varepsilon$

We want to construct a sequence in $D(f)$ that goes to p but when f is applied it does not goes to q, thus contradicting

Pick $\varepsilon > 0$ as posed before. Take $\delta_n = \frac{1}{n}$ (any positive function that goes to zero will work).
$n = 1, 2, 3, \ldots$ For each n, there exists $p_n \in D(f)$ s.t.

$0 < d_X(p_n, p) < \delta$ and $d_Y(f(p_n), q) \geq \varepsilon$.

therefore the sequence $\{p_n\}$, with n s.t. $\delta_n = \frac{1}{n}$ is s.t. $p_n \to p$, $p_n \neq p$ but $f(p_n) \not\to q$ since $d_Y(f(p_n), q) \not\to 0$. A contradiction.

ALGEBRAIC OPERATIONS: If we restrict to the case where Y is a field then if $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$ then

(a) $\lim_{x \to p} (f+g)(x) = A + B$

(b) $\lim_{x \to p} (fg)(x) = AB$

(c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

Proof follows from analogous properties of sequences, using the characterization of limit as limit of sequences.

CONTINUOUS FUNCTIONS:

Definition: (X, d_X), (Y, d_Y) metric spaces. $D(f) \subset X$, $p \in D(f)$; $f: D(f) \to Y$.
f is continuous at p if: $\forall \varepsilon > 0 \; \exists \delta > 0$: $d_Y(f(x), f(p)) < \varepsilon$ whenever $d_X(x, p) < \delta$.
If f is continuous at every point of $D(f)$, then f is said to be continuous on $D(f)$.
In this case f has to be defined at the point p in order to be continuous at p.

CHARACTERIZATION: f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Another characterization: In terms of sequences, f is continuous at p if:

(a) $\lim_{p_n \to p} f(p_n) = q$

(b) $f(p)$ is defined

(c) $f(p) = \lim_{p_n \to p} f(p_n) = q$ (the true value matches the expected value)

theorem: Composition of continuous functions yields a continuous function.
Suppose X, Y, Z are metric spaces.

$f: X \to Y$; $g: Y \to Z$ $(f: D(f) \to Y$, $g: f(D(f)) \to Z)$.

$h: X \to Z$; $(h: D(f) \to Z)$. h defined by

$h(x) = g(f(x))$ If f is continuous at $p \in D(f)$ and if g is continuous at $f(p)$

Pf. We have that:

$$\text{Given } \varepsilon > 0: \begin{cases} \exists \delta_1 > 0 : \text{If } d_X(x,p) < \delta_1 \text{ then } d_Y(f(x), f(p)) < \varepsilon \\ \exists \delta_2 > 0 : \text{If } d_Y(\underset{f(x)}{f(p)}, \overset{}{y}) < \delta_2 \text{ then } d_Z(g(f(p)), \underset{g(f(x))}{g(y)}) < \varepsilon. \end{cases}$$

We want to show that

$$\text{Given } \varepsilon > 0 : \begin{cases} \exists \delta_3 > 0 : \text{If } d_X(x,p) < \delta_3 \text{ then } d_Z(h(x), h(p)) < \varepsilon \\ \iff d_Z(g(f(x)), g(f(p))) < \varepsilon \end{cases}$$

BEGIN: Let $\varepsilon > 0$. Pick $\delta_1 > 0$. Suppose that:

$d_Y(f(p), f(x)) < \delta_1 \Rightarrow d_Z(g(f(p)), g(f(x))) < \varepsilon$.

For $\delta_1 > 0$ there exists $\delta_2 > 0$ such that.

$d_X(x,p) < \delta_2 \Rightarrow d_Y(f(x), f(p)) < \delta_1$.

Pick $\delta_2 > 0$ s.t. $d_X(x,p) < \delta_2 \Rightarrow d_Y(f(x), f(p)) < \delta_1 \Rightarrow d_Z(g(f(p)), g(f(x))) < \varepsilon$.

$\Rightarrow d_X(x,p) < \delta_2 \Rightarrow d_Z(g(f(p)), g(f(x))) < \varepsilon \iff d_Z(h(p), h(x)) < \varepsilon$.

Given $\varepsilon > 0$, pick $\delta_2 > 0$ as before and the result holds

THEOREM (characterization of continuity by open sets).

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f: X \to Y$.

f continuous at X $\iff \forall V \subset Y$, V open : $f^{-1}(V)$ is open in X.

$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$.

Note: direct image of open sets say nothing about continuity. For example the function $f(x) = 1$ is continuous, but $f((a,b)) = \{1\}$, so an open set gets mapped into a closed set.

Pf: ($\Rightarrow$) Suppose that f is continuous on X and V is an open set in Y. We want to show that every point of $f^{-1}(V)$ is interior to $f^{-1}(V)$.

i.e., $\forall p \in f^{-1}(V)$ there exists $r > 0$ s.t. $N_r(p) \subset f^{-1}(V)$.

Let $p \in f^{-1}(V)$. (Note that if there is no such p, then $f^{-1}(V) = \emptyset$, which is open and we are done. So suppose there is such a p). Since $p \in f^{-1}(V)$ we have that $f(p) \in V$. But V is open, therefore there exists $\varepsilon > 0$ s.t. $N_\varepsilon(f(p)) \subset V$, which means that $f^{-1}(N_\varepsilon(f(p))) \subset f^{-1}(V)$.

Since f is continuous, pick $\delta > 0$ for this ε s.t. If $d_X(x,p) < \delta$ then $d_Y(f(x), f(p)) < \varepsilon$. Consider $N_\delta(p) := \{x \in X : d_X(x,p) < \delta\}$. For any point $n \in N_\delta(p)$, We have that $d_X(x,p)<\delta \Rightarrow d_Y(f(x),f(p))<\varepsilon$. by continuity of f.

$\Rightarrow f(x)\in N_\varepsilon(f(p)) \subset V \Rightarrow N_\delta(p)\subset f^{-1}(V)$

Hence, for $p\in f^{-1}(V)$ we have found $\delta>0$ s.t $N_\delta(p)\subset f^{-1}(V)$, so every point of $f^{-1}(V)$ is interior, wich means that $f^{-1}(V)$ is open.

($\Leftarrow$) Suppose that $\forall V\subset Y$, V open $f^{-1}(V)$ is open in X. We want to show that f is continuous.

Fix $p\in X$ and $\varepsilon>0$. Let $V=\{y\in Y: d_Y(y,f(p))<\varepsilon\}=N_\varepsilon(f(p))$.

Clearly V is open and by assumption $f^{-1}(V)$ is open in X.

So, for any $x\in f^{-1}(V)$, there exists $\delta>0$ s.t. $N_\delta(x)\subset f^{-1}(V)$

If $x\in f^{-1}(V)$, then $f(x)\in V \Rightarrow d_Y(f(x),f(p))<\varepsilon$. ∴ so f is continuous by the δ-ε definition of continuity.

<u>Corollary</u>: $f:(X,d_X)\to(Y,d_Y)$ is continuous iff $\forall C\in Y$, C closed, $f^{-1}(C)$ is closed in X.

<u>Pf</u>: By previous theorem f continuous $\Leftrightarrow$ $f^{-1}(V)$ open $\forall$ V open in Y. Use this together with A set C is closed iff C^c is open

$f^{-1}(E^c)=[f^{-1}(E)]^c$ $\forall E\subset Y$. Let us prove this.

($\subseteq$) let $x\in f^{-1}(E^c)$, then $f(x)\in E^c \Rightarrow f(x)\notin E \Rightarrow x\notin f^{-1}(E)$ $\Rightarrow x\in[f^{-1}(E)]^c$.

($\supseteq$) Let $x\in[f^{-1}(E)]^c \Rightarrow x\notin f^{-1}(E) \Rightarrow f(x)\notin E \Rightarrow f(x)\in E^c \Rightarrow x\in f^{-1}(E^c)$

($\Rightarrow$) f continuous $\Rightarrow f^{-1}(V)$ is open $\Rightarrow [f^{-1}(V)]^c$ is closed

$\Rightarrow [f^{-1}(V)]^c=f^{-1}(V^c)$ is closed ; since V open $\Rightarrow V^c$ is closed so inverse image of closed set is closed.

($\Leftarrow$) $f^{-1}(C)$ is closed $\Rightarrow [f^{-1}(C)]^c$ is open $\Rightarrow f^{-1}(C^c)$, open, C^c open

$\Rightarrow$ by previous theorem f is continuous.

How does continuity interact with compactness?

For example, $f(x)=\frac{x^2}{x^2+1}$ $\quad f(\mathbb{R})=[0,1)$ not compact

THEOREM: Let (X,d_X), (Y,d_Y) be metric spaces. Let $f: X \to Y$.

If: (a) X is compact
(b) f is continuous.

then: $f(X)$ is compact.

Two different proofs: (I) direct (using def of compactness, (II) by sequences.

(I) We want to show that any open cover of $f(X)$; i.e., $f(X) \subset \bigcup_\alpha V_\alpha$, where V_α is open $\forall \alpha$, contains a finite subcover, $f(X) \subset \bigcup_{i=1}^n V_{\alpha_i}$, for some indices $\alpha_1, \dots, \alpha_n$.

Let $\{V_\alpha\}$ be an open cover of $f(X)$: $f(X) \subset \bigcup_\alpha V_\alpha$. Apply f^{-1}

$f^{-1}(f(X)) \subset f^{-1}(\bigcup_\alpha V_\alpha) \iff X \subset \bigcup_\alpha f^{-1}(V_\alpha)$.

By previous theorem, since f is continuous and V_α is open $\Rightarrow$ $f^{-1}(V_\alpha)$ is open $\forall \alpha$. therefore $\{f^{-1}(V_\alpha)\}$ form an open cover of X. But X is compact, so any open cover contains a finite subcover, in particular $\{f^{-1}(V_\alpha)\}$. thus $X \subset \bigcup_{i=1}^n f^{-1}(V_{\alpha_i})$, finally, apply f to both sides $f(X) \subset f(\bigcup_{i=1}^n f^{-1}(V_{\alpha_i})) \iff f(X) \subset \bigcup_{i=1}^n V_{\alpha_i}$, so we have found a finite subcover of $f(X) \Rightarrow f(X)$ is compact.

(II) the idea here is to use the theorem (proved later) that says:
X is compact $\iff$ X is sequentially compact ($\forall \{x_n\} \subset X$, $\exists \{x_{n_k}\}$ and $x \in X$ s.t. $x_{n_k} \to x$)

So take a sequence $\{q_n\} \subset f(X)$. W.T.S that it has a convergent subsequence. Since $q_n \in f(X) \Rightarrow q_n = f(p_n)$ for $p_n \in X$ ($\forall n$). Now, $\{p_n\}$ is a sequence in X, i.e., $\{p_n\} \subset X$. But X is compact. therefore X is sequentially compact so there exists a subsequence $\{p_{n_e}\}$ that converges to a point $x \in X$. Look at $f(p_{n_e}) = q_{n_e}$, so we obtain $\{q_{n_e}\}$ a subsequence of $\{q_n\}$. But f is continuous and by characterization of continuity by sequences: Since $\lim_{n_e \to \infty} p_{n_e} = x$ we have that $\lim_{n_e \to \infty} f(p_{n_e}) = f(x)$, so $q_{n_e} \to f(x)$.

So we have found for an arbitrary sequence $\{q_n\} \subset f(X)$ a convergent subsequence $\{q_{n_e}\}$ that converges to a point $f(x) \in f(X)$.

This means that X is sequentially compact, which in metric spaces is equivalent to X being compact.

Application: (X, d_X), Let $f: X \to \mathbb{R}$ Suppose: X compact and f continuous

ⓐ $f(X)$ is compact $\Leftrightarrow$ $f(X)$ closed and bounded.

ⓑ Let $M = \sup\{f(p) : p \in X\}$. Then, $\exists\, x_M \in X$ s.t. $f(x_M) = M$

ⓒ Let $m = \inf\{f(p) : p \in X\}$. Then, $\exists\, x_m \in X$ s.t. $f(x_m) = m$

ⓑ & ⓒ $\Leftrightarrow$ $\exists$ points M & $m \in X$ s.t. $f(m) \leq f(x) \leq f(M)$ $\forall x \in X$

that is f attains its maximum at M and its minimum at m.

Pf: ⓐ follows from theorem that in $\mathbb{R}$ compact $\Leftrightarrow$ closed & bounded.

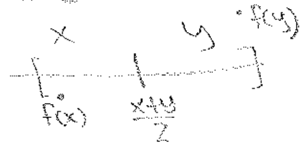
ⓑ & ⓒ $f(X) \subset \mathbb{R}$ compact $\Rightarrow$ closed & bounded. From bounded we get that sup and inf exists. From closed we get that $M \in f(X)$ and $m \in f(X)$. From properties of sup and inf $f(m) \leq f(x) \leq f(M)$, $\forall x \in X$

Intermediate value theorem:

Let $f: I \to \mathbb{R}$, where f is continuous and I is a closed interval

If there exists $x, y \in I$ such that $f(x) < f(y)$ and $f(x) < L < f(y)$

then there exists $z \in I$ such that $f(z) = L$.

Pf: 

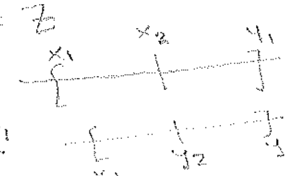
Define sequences $\{x_n\}$ and $\{y_n\}$ as follow:

$x_1 = x$

$y_1 = y$

. Now, depending on the value of $f(\frac{x+y}{2})$:

$$f\left(\frac{x+y}{2}\right) = \begin{cases} = L, & \text{we are done, } \frac{x_1+y_1}{2} = z \\ < L & \Rightarrow x_2 = \frac{x_1+y_1}{2}\ ;\ y_2 = y_1 \\ > L & \Rightarrow x_2 = x_1\ ,\ y_2 = \frac{x_1+y_1}{2} \end{cases}$$



By CANTOR'S THEOREM, since the interval decreases, $\{x_n\}, \{y_n\}$ have a common limit point, z say. Then, since f is continuous

$$\lim_{n\to\infty} x_n = z = \lim_{n\to\infty} y_n \xRightarrow[\text{cont.}]{f} \lim_{n\to\infty} f(x_n) = f(z) = \lim_{n\to\infty} f(y_n).$$

But by construction of the sequences:

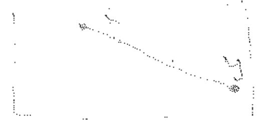
$$\lim_{n\to\infty} f(y_n) \geq L \Leftrightarrow f(z) \geq L \quad \text{and} \quad \lim_{n\to\infty} f(x_n) \leq L \Leftrightarrow f(z) \leq L$$

$$\Rightarrow \quad f(z) \geq L \quad \text{and} \quad f(z) \leq L$$

therefore, by properties of ordered field of real numbers

$$f(z) = L.$$

Note that this proof of the I.V.P holds in $\mathbb{R}^n$. For example, in $\mathbb{R}^2$

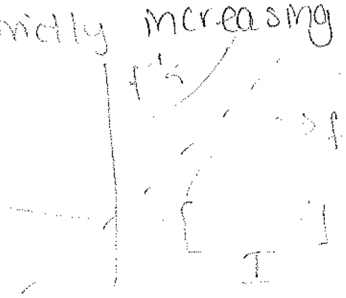


Join x and y by a line. Then you are back to the 1D case.

Inverse function

THEOREM [on $\mathbb{R}$] $f: I \to \mathbb{R}$ be a 1-1, continuous map, let I be a closed interval then $f^{-1}: f(I) \to I$ is continuous.

Pf: (Sketch). FACT: f is monotone (b/c it is continuous and 1-1). Moreover, f is strictly increasing or decreasing.



To get the graph of the inverse, reflect on the line $y=x$. You can see f^{-1} has to be continuos

(In general metric spaces).

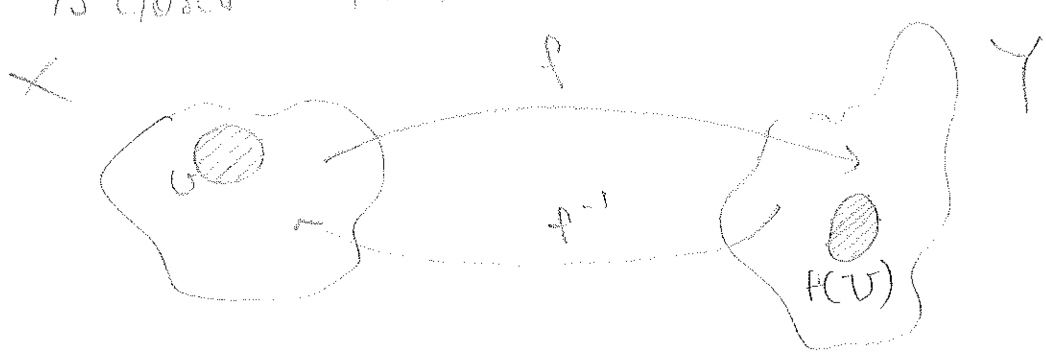
THEOREM: let (X, d_X), (Y, d_Y) be metric spaces X compact, f continuous and 1-1. Then f^{-1} defined as $f^{-1}(f(x)) = x \quad \forall x \in X$ is well-defined and is continuous.

Pf: that f^{-1} is well-defined follows directly from f being 1-1.

Let us prove that f^{-1} is continuous by open set characterization, i.e., we want to show that for every open set $U \subset X$: $(f^{-1})^{-1}(U)$ is open, $\iff f(U)$ is open in Y.

Let U be open in X. Then $U^c = X \setminus U$ is closed. Now, $U^c \subset X$ and X is compact $\Rightarrow U^c$ is compact.

Now, since f is continuous, $f(U^c)$ is compact in Y. Hence, $f(U^c)$ is closed. Since f is 1-1 $f(U^c) = f(U)^c$ so that $f(U)^c$ is closed $\Rightarrow f(U)$ is open in Y.



Uniformly Continuous: (does not depend on the point p being approached).

Def: Let (X, d_X) and (Y, d_Y) be metric spaces. We say that f is uniformly continuous on X if:

$\forall \varepsilon > 0 : \exists \delta > 0 : d_Y(f(p), f(q)) < \varepsilon$ for all $p, q \in X$ for which $d_X(p, q) < \delta$

Note that the main difference between continuity and uniform continuity is that continuity is a notion on a single point whereas uniform continuity is a notion in a set.

f continuos on $X \Rightarrow$ Given $\varepsilon > 0$ find $\delta = \delta(p, \varepsilon, f)$. whereas

f uniformly continuous on $X \Rightarrow$ Given $\varepsilon > 0$ find $\delta = \delta(\varepsilon, f)$.

FACT: If f is uniformly continuous $\Rightarrow$ f is continuous.

COMPACTNESS - Continuity: Q: when is f uniformly continuous?

Definition: We say that $A \subset X$ is sequentially compact If for any $\{a_n\} \subset A$, there exists $\{a_{n_k}\}$ and $a \in A$ s.t. $a_{n_k} \to a$.

THEOREM: If f is continuous, $f: A \to Y$, where A is sequentially compact then f is uniformly continuous.

Pf: By contradiction. Suppose A is sequentially compact and f is cont. But f is not uniformly compact. then

$\exists \varepsilon > 0 : \forall \delta > 0 : \exists a, a' \in A$ $(a \neq a')$ for which $d_X(a, a') < \delta$ but $d_Y(f(a), f(a')) \geq \varepsilon$

Pick such $\varepsilon > 0$.

Now, pick a sequence of $\{\delta_n\}$ going to zero, say $\delta_n = \frac{1}{n}$. Consider sequences: $\{a_n\}, \{a_n'\}$: $d_X(a_n, a_n') < \delta_n = \frac{1}{n}$ but $d_Y(f(a_n), f(a_n')) \geq \varepsilon$, $\forall n$.

Observe that $\{a_n\} \subset A$ and A is seq. compact, so $\exists \{a_{n_k}\}$ a sub. seq of $\{a_n\}$ and $\exists a \in A$ s.t $a_{n_k} \to a$. Moreover $\{a_{n_k}'\}$ also converges to a because

$$d_X(a_{n_k}', a) \leq \underbrace{d_X(a_{n_k}', a_{n_k})}_{\to 0} + \underbrace{d(a_{n_k}, a)}_{\to 0}.$$

But then $d_Y(f(a_{n_k}), f(a_{n_k}')) \geq \varepsilon^{(*)}$. But f is continuous, so by continuity by sequences.

$f(a_{n_k}) \to f(a)$, $f(a_{n_k}') \to f(a)$. A contradiction because we have two sequences going to the same limit, so the differences has to go to zero but they don't by (*). Hence, f is uniformly continuous

When is a set sequentially compact? (We will see compact $\Leftrightarrow$ seq compact).

Ex: Let X be a set of real sequences. So $x \in X \Leftrightarrow x = (x_1, x_2, \dots), x_i \in \mathbb{R}$.
Let us define the distance: $d_X(x,y) = \sup_n |x_n - y_n| < \infty$. We can show that (X, d_X) is a complete metric space.
Now, consider the sequence in X: $e_n = (\overbrace{0, \dots, 0}^{n-1}, 1, 0, 0, \dots)$. (unit vector) infinite dimensional.
Define the set $A = \{e_n : n = 1, 2, \dots\} \subset X$. Note that $d_X(e_n, e_m) = 1$ if $n \neq m$.
A is closed and bounded: A is closed because there are no limit points.
A is bounded because $A \subset$ unit ball at origin (infinite dim. ball).
But A is not compact because if you choose the open cover $\{N_1(e_n)\}$, $n = 1, 2, 3, \dots$ It has no finite subcover, you need all $N_1(e_n)$.
In particular A is not sequentially compact.

Definition: A subset A of a metric space (X, d_X); $A \subseteq X$, is said to be totally bounded if $\forall \varepsilon > 0$, $\exists$ finitely many points $a_1, \dots, a_n \in A$ s.t.

$$A \subset \bigcup_{m=1}^{n} N_\varepsilon(a_m). \quad (\{N_\varepsilon(a_m)\}_{n=1}^{m} \text{ is often called } \varepsilon\text{-net})$$

Ex (I) Consider the half open interval $(0,1] \subset \mathbb{R}$ $\quad (\quad]$ this set is totally bounded; but we know $(0,1]$ is not compact.
totally bounded $\not\Leftrightarrow$ compact. (We also need complete $\approx$ closed).
(II) in our previous example i.e., X: real sequences. The set $A = \{e_n ; n = 1, 2, \dots\}$ is not totally bounded because: $\cancel{(e_1)(e_2)(e_3)}$ ... So there is no finite ε-net cover.
Note that $A \subset X$, A not totally bounded $\Rightarrow$ X is not totally bounded
If a smaller set is not totally bounded then clearer the bigger set can't be.

Proposition: Let (X, d) be a metric space. Let $A \subset X$ be sequentially compact then A is complete and totally bounded. (Complete = every Cauchy sequence converges)

Pf: Need to prove (I) A is complete (II) A is totally bounded.
(I) Let $\{a_n\} \subset A$ be a Cauchy sequence. N.T.P $\{a_n\}$ converges.
Since A is sequentially compact, $\exists \{a_{n_k}\} \subset A$ and $a \in A$ s.t. $a_{n_k} \to a$ as $n_k \to \infty$.
We proved that if a Cauchy sequence has a convergent subsequence then it must converge (actually to the same limit). So $a_n \to a$ as $n \to \infty$.
So, every Cauchy sequence in A converges. Thus, A is complete.
(II) Want to show that A is totally bounded given that A is

Given $\varepsilon > 0$, let $a \in A$. Look at $N_\varepsilon(a)$ Define the sequence $\{a_n\}$ as follows

If $A \subset N_\varepsilon(a)$ then we are done.
O/w, relabel $a = a_1$ and pick $a_2 \in A \setminus N_\varepsilon(a_1)$ which is guaranteed to exists.

If $A \subset N_\varepsilon(a_1) \cup N_\varepsilon(a_2)$ then we are done.
O/w pick $a_3 \in A \setminus N_\varepsilon(a_1) \cup N_\varepsilon(a_2)$ which is guaranteed to exists.

$\vdots$

Having picked $a_1, \ldots, a_k$, consider:
If $A \subset \bigcup_{l=1}^{k} N_\varepsilon(a_l)$ then we are done
O/w pick $a_{k+1} \in A \setminus \bigcup_{l=1}^{k} N_\varepsilon(a_l)$, keep doing this process.
We want to show that this process stop, i.e., $\{a_n\}$ is finite
Suppose, for a contradiction, that the process goes on forever.
Observe that $d(a_m, a_n) \geq \varepsilon$ (*) because, by construction, $a_2 \notin N_\varepsilon(a_1)$,
$a_3 \notin N_\varepsilon(a_2)$ and $a_3 \notin N_\varepsilon(a_2)$, and so on.
Now, assuming $\{a_n\}$ is infinite, since A is sequentially compact, we know
there exists $\{a_{n_k}\}$ and $a \in A$ s.t $a_{n_k} \to a$ as $n_k \to \infty$. But if this is the
case then $d(a_{n_{k_l}}, a_{n_{k_m}}) < \varepsilon$, a contradiction with (*). which means that
therefore, the process stop and $\{a_n\}$ is finite
$A \subset \bigcup_{l=1}^{m} N_\varepsilon(a_l)$, for any choice of $\varepsilon > 0$. So A is totally bounded.

<u>THEOREM</u> Let (X, d) be a metric space. Let $A \subset X$.
A is compact $\iff$ A is sequentially compact

Pf:
($\Rightarrow$) Suppose A is compact. Take a sequence $\{a_n\} \subset A$. Look at the set
$S = \{a_1, a_2, \ldots\} \subset A$. Since A is compact and compact sets are closed,
A is closed so it contains all of its limit points, i.e., $\overline{A} = A$.
therefore $\overline{S} \subset \overline{A}$. So any limit point of S is in A.
($\Leftarrow$) Suppose A is sequentially compact. We will prove next that this imply
that A is complete and totally bounded. Need to prove A is compact.
By contradiction, suppose that there exists $\mathcal{C}$, a cover of A by open sets,
which has no finite subcover. Since A is totally bounded, let $\alpha = \dim(A) < \infty$
and there exists finitely many $\{a_1, \ldots, a_l\}$ s.t. $A \subset \bigcup_{j=1}^{l} N_{\alpha/4}(a_j)$.

C is open
so x is interior

$\mathcal{C}$ is an open covering of each $A \cap N_{\frac{\alpha}{4}}(a_j)$, for $1 \le j \le l$
Pick $A \cap N_{\frac{\alpha}{4}}(a_j)$ s.t. it has no finite subcover.
Consider $x_1 \in A \cap N_{\frac{\alpha}{4}}(a_j)$. $\mathcal{C}$ does not finitely cove the intersection
For $\frac{\alpha}{4^2}$, pick x_2 s.t. the same as before is true. Pick $x_3, x_4, \dots$, like this
then $\{x_n\} \subset A$, and $\mathcal{C}$ does not finitely cover $\{x_n\}$.
But A is sequentially compact, so $\exists \{x_{n_k}\} \subset A$ and $x \in A$ s.t. $x_{n_k} \to x$

has no finite subcover

But then $C \in \mathcal{C}$ is a finite subcover of $\{x_n\}$.
Contradiction. Therefore A is compact.

Important theorem in METRIC SPACES:

$\boxed{\text{Theorem: } A \text{ compact iff } A \text{ is complete (closed) \& totally bounded}}$

Pf (sketch)
($\Longrightarrow$) Use same idea as in previous proof
($\Longleftarrow$) A complete & totally bounded. $\{x_n\}$ as in previous proof. Then $\{x_n\}$
is Cauchy (diameter goes to zero). Then $x_n \to x$, where this x is the
same as in previous proof (Again, strategy as before by contradiction).

Diagram: Continuity metric spaces

A compact $\Longleftrightarrow$ A is sequentially compact
$\Updownarrow$
A is complete (closed) & totally bounded

$f: (X, d_X) \to (Y, d_Y)$, continuous,
A sequentially compact (or compact, or complete and totally bounded) then
f is uniformly continuous

Examples: Let us explore the various theorems we have proved.
What if f is continuous but A is not compact?
$f: A \to \mathbb{R}$, $A \subset \mathbb{R}$. What if f is continuous
If A is not compact then: $\begin{cases} A \text{ is not bounded} : f(x) = x \\ A \text{ is not closed } (*) \end{cases}$

$(*)$ $\exists x_0 \in \bar{A} \setminus A$. define:

$f(x) = \dfrac{1}{x - x_0}$ $(x \in A)$. Then f is continuous (composition of continuous is continuous) and f is not bounded.
this function is not uniformly continuous.