M436 - Introduction to Geometries - Homework 10

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(Ex. 1) Find the center and radius of the circle through the points $p_1 = (2/5, 1/5)$ and $p_2 = (3/5, -2/5)$ that is perpendicular to the unit circle.

Solution: From previous homework we know that the angle of intersection ϕ between two circles is given by:

$$cos(\phi) = \frac{r^2 + s^2 - |p-q|^2}{2rs} \quad \text{, where } r, s > 0 \text{ are the radii of the circles and } p, q \in \mathbb{R}^2 \text{ are the centers}$$

Take one of the circles to be the unit circle and the other to be our unknown circle. Then, $p = (0,0), r = 1$, $q = (q_1, q_2)$ and $s = s$. We want $\phi = \pi/2$. This implies that $cos(\phi) = 0$. Hence:

$$0 = \frac{1^2 + s^2 - |(0,0) - (q_1, q_2)|^2}{2 \cdot 1 \cdot s} \Longrightarrow 0 = 1 + s^2 - (q_1^2 + q_2^2) \Longrightarrow q_1^2 + q_2^2 = 1 + s^2$$

We also know that the circle with radius s and center (q_1, q_2), passes through the points $(2/5, 1/5)$ and $(3/5, -2/5)$. In other words, these points satisfy the equations:

$$(2/5 - q_1)^2 + (1/5 - q_2)^2 = s^2 \quad \Longrightarrow \quad \tfrac{1}{5} + q_1^2 + q_2^2 - \tfrac{4}{5}q_1 - \tfrac{2}{5}q_2 = s^2$$

$$(3/5 - q_1)^2 + (-2/5 - q_2)^2 = s^2 \quad \Longrightarrow \quad \tfrac{13}{25} + q_1^2 + q_2^2 - \tfrac{6}{5}q_1 + \tfrac{4}{5}q_2 = s^2$$

Substract from the first equation the second equation to obtain:

$$\frac{5}{25} - \frac{13}{25} - \frac{4}{5}q_1 + \frac{6}{5}q_1 - \frac{2}{5}q_2 - \frac{4}{5}q_2 = 0 \Longrightarrow -\frac{8}{5} + 2q_1 - 6q_2 = 0 \Longrightarrow q_1 = \frac{4}{5} + 3q_2$$

Now we can find q_2 by replacing q_1 in terms of q_2 as follows:

$$\frac{1}{5} + q_1^2 + q_2^2 - \frac{4}{5}q_1 - \frac{2}{5}q_2 = s^2 \Longrightarrow \frac{1}{5} + \left(\frac{4}{5} + 3q_2\right)^2 + q_2^2 - \frac{4}{5}\left(\frac{4}{5} + 3q_2\right) - \frac{2}{5}q_2 = s^2$$

$$q_1^2 + q_2^2 = 1 + s^2 \Longrightarrow \left(\frac{4}{5} + 3q_2\right)^2 + q_2^2 = 1 + s^2$$

Finally, replace s from the first equation into the second:

$$\left(\frac{4}{5} + 3q_2\right)^2 + q_2^2 = 1 + \frac{1}{5} + \left(\frac{4}{5} + 3q_2\right)^2 + q_2^2 - \frac{4}{5}\left(\frac{4}{5} + 3q_2\right) - \frac{2}{5}q_2 \Longrightarrow$$

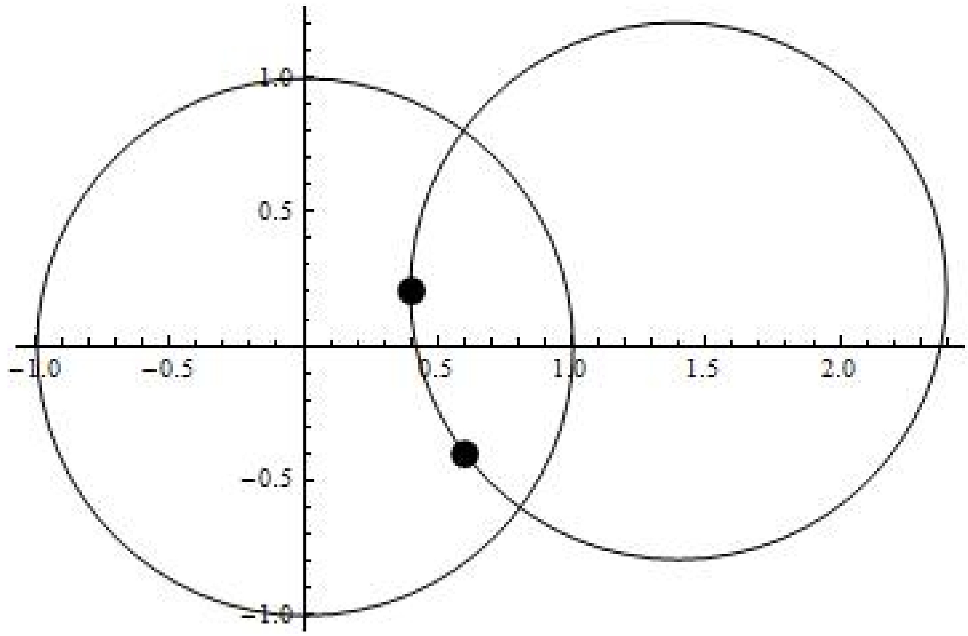
$$0 = 1 + \frac{1}{5} - \frac{16}{25} - \frac{12}{5}q_2 - \frac{2}{5}q_2 \Longrightarrow 0 = \frac{6}{5} - \frac{16}{25} - q_2\left(\frac{12}{5} + \frac{2}{5}\right) \Longrightarrow \frac{14}{5}q_2 = \frac{14}{25} \Longrightarrow q_2 = \frac{5}{25} \Longrightarrow \boxed{q_2 = \frac{1}{5}}$$

We can solve for q_1 by plugging the value for q_2 into $q_1 = \frac{4}{5} + 3q_2 \Longrightarrow q_1 = \frac{4}{5} + \frac{3}{5} \Longrightarrow \boxed{q_1 = \frac{7}{5}}$.

Solve for the radius s from the equation: $q_1^2 + q_2^2 = 1 + s^2 \Longrightarrow \left(\frac{7}{5}\right)^2 + \left(\frac{1}{5}\right)^2 = 1 + s^2 \Longrightarrow s^2 = \frac{49}{25} + \frac{1}{25} - 1 \Longrightarrow s^2 = 2 - 1$, and so $s = \pm 1$, but the radius must be positive and hence, $\boxed{s = 1}$.

In conclusion, the center of the circle through the points $p_1 = (2/5, 1/5)$ and $p_2 = (3/5, -2/5)$ that is perpendicular to the unit circle is $\boxed{(7/5, 1/5)}$ and the radius is $\boxed{1}$.

The following graph depicts this solution.



(Ex. 2) Find a hyperbolic isometry of the upper half plane that fixes $(0,1)$ and that rotates every geodesic through $(0,1)$ into a geodesic perpendicular to the original one. Write the isometry as a Mobius transformation.

Solution: Let ϕ be the Mobius transformation we wish to find. We note that the following mappings must take place:

$$\phi(-1) = 0, \quad \phi(0) = 1, \quad \phi(1) = \infty, \quad \phi(i) = i$$

We also know the general form of a Mobius transformation: $\phi(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$. Plugging in we have:

$$\phi(0) = 1 = \frac{a \cdot 0 + b}{c \cdot 0 + d} = \frac{b}{d} \Longrightarrow b = d \quad \text{, from this point on we have: } \phi(z) = \frac{az+b}{cz+b}$$

$$\phi(1) = \infty = \frac{a \cdot 1 + b}{c \cdot 1 + b} = \frac{a+b}{c+b} \Longrightarrow c + b = 0 \Longrightarrow c = -b \quad \text{, hence: } \phi(z) = \frac{az+b}{-bz+b}$$

$$\phi(-1) = 0 = \frac{a \cdot (-1) + b}{-b \cdot (-1) + b} = \frac{-a+b}{2b} \Longrightarrow -a + b = 0 \Longrightarrow a = b \quad \text{, thus: } \phi(z) = \frac{az+b}{-bz+b}$$

$$\phi(i) = i = \frac{a \cdot i + b}{-b \cdot i + b} \Longrightarrow i(-bi+b) = ai + b \Longrightarrow b + bi = ai + b \Longrightarrow bi = ai \Longrightarrow a = b \quad \text{,we have: } \phi(z) = \frac{bz+b}{-bz+b}$$

Finally, by algebra: $\phi(z) = \dfrac{bz+b}{-bz+b} = \dfrac{b(z+1)}{b(-z+1)} = \dfrac{z+1}{-z+1}$. So the hyperbolic isometry ϕ of the upper half plane that fixes $(0,1)$ and that rotates every geodesic through $(0,1)$ as a Mobius transformation is given by:

$$\boxed{\phi(z) = \frac{z+1}{-z+1}}$$

One can easily check that indeed this isometry fixes i:

$$\phi(i) = \frac{1+i}{1-i} = \frac{i+1}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+2i+i^2}{1^2-i^2} = \frac{2i}{2} = i$$

Since Mobius transformation preserve angles, the definition of ϕ will be enough to rotate every geodesic through $(0,1)$ into a geodesic perpendicular to the original one.

(Ex. 3) Given two circles in the plane that have no point in common, show that there is a Mobius transformation that takes the two circles to concentric circles.

Solution: Let C_1 and C_2 be two circles in the plane that have no point in common of radii r_1 and r_2, and centers z_1 and z_2 respectively. Without loss of generality let us transform one of these circles, say C_1, into the unit circle. For this, we can defined the following Mobius transformation:

$$\phi_1(z) := \frac{z - z_1}{r_1}$$

Note that the image of C_1 under ϕ_1 is the unit circle. This is true because we know that Mobius transformation map circles to circles. Moreover, we can see that $\phi_1(z_1) = (0, 0)$ so that the center of C_1 is mapped to the origin in the complex plane and the radius is shrink by a factor of r_1 to normalize it to the unit circle.
Now, the image of the second circle C_2 under ϕ_1 is also a circle but in this case of center $z_2 - z_1$ and some radius, call it r_2'. At this point we have two cases:

(i) If $|z_2 - z_1| < 1$ then the circles are already nested and we don't need to use ϕ_2, which will be defined next. (Alternative, you can think of ϕ_2 to be defined as $\phi_2(z) = z$ in this case).

(ii) Otherwise, if $|z_2 - z_1| \geq 1$, then we need to nest the circles. For that, we will define the map $\phi_2(z) := \frac{1}{z}$

Clearly, in any case the image under $\phi_2 \circ \phi_1$ of C_1 is still the unit circle since the inversion $1/z$ fixes the unit circle. The second circle C_2 is now a circle centered at $\frac{1}{z_1 - z_0}$ with some radius r_2''.

At this point we have produced two nested circles out of our initial circles C_1 and C_2. To finish we would need to translate the center of C_2 to the origin to obtain two concentric circles. For this, let us define a final Mobius transformation ϕ_3 as follow:

$$\phi_3(z) := \frac{z - (z_2 - z_1)}{1 - \overline{(z_2 - z_1)} z}$$

Therefore, the image under $\phi_3 \circ \phi_2 \circ \phi_1$ of C_1 is the unit circle and the image of C_2 is a circle centered at $(0, 0)$ with some radius say r_2''' An almost complete formula (assuming that C_1, C_2 are not nested -case (ii)) is given by:

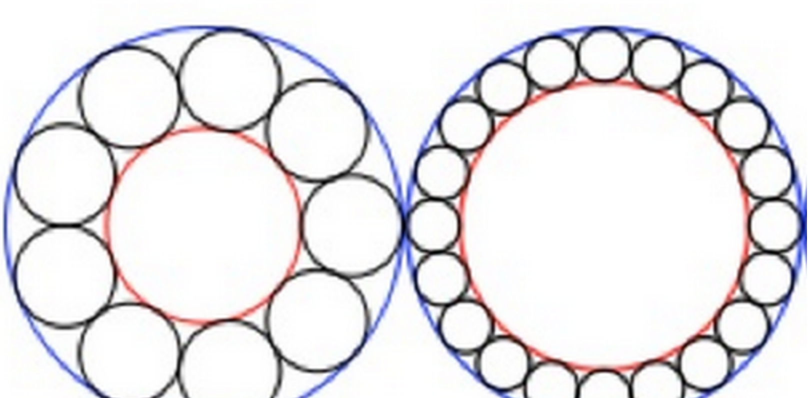
$$\phi_3 \circ \phi_2 \circ \phi_1(z) = \frac{(\frac{r_1}{z - z_1}) - (z_2 - z_1)}{1 - \overline{(z_2 - z_1)}(\frac{r_1}{z - z_1})}$$

Note that the radii r_2, r_2', and r_2'' could be defined as functions of r_1 and r_2, so there is no problem with these.

(Ex. 4) Given two circles one inside the other, form a chain of circles that touch consecutively and always the inner and outer circle. Show that if this chain closes, it closes for all choices of initial circles.

Solution: Suppose you have two circles, one inside the other and you form a chain of circles that touch consecutively and always the inner and outer circle. Further suppose that this chain closes. Then, we have two cases

(a) The circles are <u>concentric</u>, then we have a situation like this one:



Clearly, a chain of circles with this structure, i.e., such that the first two circles are concentric and the chain closes could be started anywhere and it will still close. In other words, if this chain closes, it closes for all choices of initial circles. From the picture we can see that, by symmetry, any choice of placement for the initial circle will produce a closed chain.

(b) The circles are not concentric. This case can be reduced to the previous case. I will explain how: take the two initial circles. By hypothesis, one circle is inside the other meaning that they do not have any point in common. In exercise 3 we prove that given two circles in the plane that have no point in common, there exists a Mobius transformation that takes the two circles to concentric circles. Use this Mobius transformation to make the inner circle be concentric with the outer. Mobius transformation preserve angles and tangency. Therefore, the chain of closing circles will close with the two concentric circles giving a picture like the one above. Now using the same reasoning as in the previous case, we could translate any of the circles in the chain and choose it to be the initial circle. Finally, take the inverse of the Mobius Transformation, which we know it exists since with every invertible complex 2-by-2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

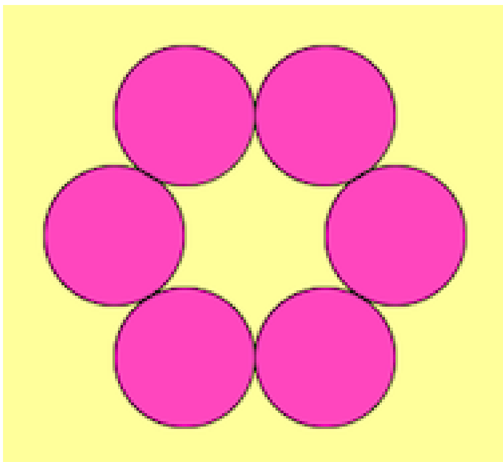
we can associate the Mobius transformation:

$$\phi(z) = \frac{az+b}{cz+d}.$$

To obtain a chain of closing circles in the original case with non concentric circles.

(Ex. 5) Start by choosing three mutually touching spheres, like the blue, rose, and translucent one in Figure 2. The translucent one is touching the others externally. Begin a chain of spheres with a sphere touching all three spheres, like the yellow sphere. Then consecutively add spheres (purple, green, pink, dark blue, moldy green) to the chain so that they touch all three initial spheres and the previously added sphere. Show that the chain of spheres always closes afters six spheres.

Solution: As discussed in class, take the initial three mutually touching spheres to hyperbolic space (I'm thinking about the upper half space model, and inverting about a given sphere). You will obtain two parallel planes and a sphere inside touching each of these planes. Now you can start placing spheres to the chain so that they touch all three initial spheres and the previously added sphere. In this plane you will immediately notice that the chain must close after six spheres because there won't be any more room for another sphere.
The following graph illustrate a cross-section of this space looked at from above:



This figure shows the six spheres that were added to the chain. The three initial sphere are: (1) a sphere in the middle (not shown in the picture), (2) the top plane and (3) the bottom plane. After doing this process, take all nine spheres back to Euclidean space to obtain the result.