# M436 - Introduction to Geometries - Homework 5

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(Ex. 1) This problem studies *ice cream geometry*, a special kind of incidence geometry. The points are a certain set of children in a class room, and the lines are ice-cream flavors. A child is incident with an ice-cream flavor if she likes that flavor.

The following axioms are known to be true:

(A1) There are exactly five flavors of ice cream: vanilla, chocolate, strawberry, cookie dough, and bubble gum.

(A2) Given any two different flavors, there is exactly one child who likes these two flavors.

(A3) Every child likes exactly two different flavors among the five.

Investigate this geometry by answering the following questions:

1. How many children are there in this classroom? Prove your result.
2. Show that any pair of children likes at most one common flavor.
3. Show that for each flavor there are exactly four children who like that flavor.

Justify all your answers.

**Solution:**

1. Let $C$ be the set of children and $F$ be the set of flavors defined as $F = \{V, C, S, D, B\}$, where $V$ stands for vanilla, $C$ for chocolate, $S$ for strawberry, $D$ for cookie dough and $B$ for bubble gum in compliance with. (A1) guarantees that $F$ contains all flavors.
   We can construct the set $C$ as follows: by axiom (A2) we know that given any two different flavors, there is exactly one child who likes these two flavors. So name a child by the two flavors that she likes, for example, one child's name would be $VC$ corresponding to the child that likes both vanilla and chocolate. Note that this name is the same as $CV$ since liking chocolate and vanilla is the same as liking vanilla and chocolate. Therefore, counting children reduces to counting different pairs of flavors that can be chosen out of $F$, which we know to be $\binom{5}{2} = \frac{5\cdot 4}{2} = 10$. This shows that $C$ contains at least 10 children.
   Finally, that $C$ contains exactly 10 children follows from (A3). Suppose there are more than 10 children. Then, any children after children 10 would have to like exactly two different flavors among five, and would necessary repeat flavors, i.e., that children was already accounted for. Hence, set $C$ is:
   $$C = \{VC, VS, VD, VB, CS, CD, CB, SD, SB, DB\}$$
2. First note that it is possible for two children not to like any flavor in common, for example child $VC$ does not like any of the flavor that child $DB$ likes. Second, axiom (A2) guarantees that no two children like the same two flavors. The only other possibility is that two children like exactly one flavor in common. In 3. there is a list that provides the children that like exactly one flavor in common, so I won't repeat it here. This list exhausts 30 possible pairs of children ($\binom{4}{2} = 6$ for each flavor) where two children like exactly one flavor in common.

   There are $(10 \cdot \binom{3}{2})/2 = 15$ pairs that do not like any flavors in common:
   $(VC, SD), (VC, SB), (VC, DB), (VS, CD), (VS, CB), (VS, DB), (VD, CS), (VD, CB), (VD, SB),$
   $(VB, CS), (VB, CD), (VB, SD), (CS, DB), (CD, SB), (CB, SD)$.

   There are $\binom{10}{2} = \frac{10\cdot 9}{2} = 45 = 30 + 15$ pair of children and we have accounted for all of them.

3. In 1. we prove that the set of children is $C = \{VC, VS, VD, VB, CS, CD, CB, SD, SB, DB\}$. Let us show that for each flavor there are exactly four children who like that flavor by listing the children who like each flavor:

   vanilla: $VC, VS, VD, VB$ chocolate: $VC, CS, CD, CB$
   strawberry: $VS, CS, SD, SB$ cookie dough: $VC, CS, CD, CB$
   bubble gum: $VB, CB, SB, DB$

   An alternative argument: choose one flavor, then there are $\binom{4}{1} = 4$ ways of completing this flavor to construct the name of a child, i.e., for each flavor there are exactly four children who like that flavor.

(Ex. 2) Show that the Möbius transformations

$$f(z) = \frac{1}{1-z} \qquad g(z) = \frac{z}{z-1}$$

generate a group of order 6 which is isomorphic to the symmetric group $S_3$ of three elements. Hint: What do these Möbius transformations do to $0, 1$, and $\infty$?

**Solution:** Let us start composing these functions to construct the group table: $f(z) = \frac{1}{1-z} \Longrightarrow f^{-1}(z) = \frac{z-1}{z}$ since

$$f \circ f^{-1}(z) = f(f^{-1}(z)) = f(\frac{z-1}{z}) = \frac{1}{1 - \frac{z-1}{z}} = \frac{1}{\frac{1}{z}} = z; \quad f^{-1} \circ f(z) = f^{-1}(f(z)) = f^{-1}(\frac{1}{1-z}) = \frac{\frac{1}{1-z} - 1}{\frac{1}{1-z}} = \frac{\frac{1}{1-z}}{\frac{1}{1-z}} = z$$

$g(z) = \frac{z}{z-1} \Longrightarrow g^{-1}(z) = \frac{z}{z-1}$ since

$$g \circ g^{-1}(z) = g(g^{-1}(z)) = g(\frac{z}{z-1}) = \frac{\frac{z}{z-1}}{\frac{z}{z-1} - 1} = \frac{\frac{z-1}{1}}{\frac{1}{z-1}} = z; \quad g^{-1} \circ g(z) = g^{-1}(g(z)) = g^{-1}(\frac{z}{z-1}) = \frac{\frac{z}{z-1}}{\frac{z}{z-1} - 1} = z$$

$$f \circ g(z) = f(g(z)) = f(\frac{z}{z-1}) = \frac{1}{1 - \frac{z}{z-1}} = \frac{1}{\frac{1}{1-z}} = 1 - z \Longrightarrow (f \circ g)^{-1} = f \circ g \text{ since}$$

$$(f \circ g) \circ (f \circ g) = f \circ g(1-z) = 1 - (1-z) = z$$

$$g \circ f(z) = g(f(z)) = g(\frac{1}{1-z}) = \frac{\frac{1}{1-z}}{\frac{1}{1-z} - 1} = \frac{\frac{1}{1-z}}{\frac{z}{1-z}} = \frac{1}{z} \Longrightarrow (g \circ f)^{-1} = g \circ f \text{ since}$$

$$(g \circ f) \circ (g \circ f) = g \circ f(\frac{1}{z}) = \frac{1}{\frac{1}{z}} = z$$

$$g \circ f^{-1} = g(f^{-1}(z)) = g(\frac{z-1}{z}) = \frac{\frac{z-1}{z}}{\frac{z-1}{z} - 1} = \frac{\frac{z}{-1}}{z} = 1 - z = f \circ g$$

$$f^{-1} \circ g = f^{-1}(g(z)) = f^{-1}(\frac{z}{z-1}) = \frac{\frac{z}{z-1} - 1}{\frac{z}{z-1}} = \frac{\frac{1}{z-1}}{\frac{z}{z-1}} = \frac{1}{z} = f \circ g$$

And in this manner, we can complete the following is the group table generated by $f$ and $g$:

| $\circ$ | $e$ | $f$ | $f^{-1}$ | $g$ | $f \circ g$ | $g \circ f$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $f$ | $f^{-1}$ | $g$ | $f \circ g$ | $g \circ f$ |
| $f$ | $f$ | $f^{-1}$ | $e$ | $f \circ g$ | $g \circ f$ | $g$ |
| $f^{-1}$ | $f^{-1}$ | $e$ | $f$ | $g \circ f$ | $g$ | $f \circ g$ |
| $g$ | $g$ | $g \circ f$ | $f \circ g$ | $e$ | $f^{-1}$ | $f$ |
| $f \circ g$ | $f \circ g$ | $g$ | $g \circ f$ | $f$ | $e$ | $f^{-1}$ |
| $g \circ f$ | $g \circ f$ | $f \circ g$ | $g$ | $f^{-1}$ | $f$ | $e$ |

Note that this group is isomorphic to $S3$ via the map: $e \mapsto ()$; $f \mapsto (123)$; $f^{-1} \mapsto (132)$; $g \mapsto (12)$, $f \circ g \mapsto (13)$; $g \circ f \mapsto (23)$, where we have three elements of order 2 that corresponds with all transpositions and two elements of order 3 that correspond to cycles of length 3.

(Ex. 3) Let $\mathbb{F}P^2$ be a projective plane. A set of four points $p_1, p_2, p_3, p_4$ is a *projective frame* if no three of them are collinear. Suppose we have two projective frames $p_1, p_2, p_3, p_4$ and $q_1, q_2, q_3, q_4$. Show that there is a projective linear transformation that maps $p_i$ to $q_i$ for $i = 1, \ldots, 4$.

**Solution:** Let us first map $(1:0:0),(0:1:0),(0:0:1)$ and $(1:1:1)$ to $p_1,p_2,p_3$ and $p_4$ respectively. As no three points are collinear, the vectors $p_1,p_2$ and $p_3$ are linearly independent and form a basis of $\mathbb{F}^3$. Hence we can write:

$$p_4 = ap_1 + bp_2 + cp_3, \qquad \text{with coefficients } a,b,c \in \mathbb{F}$$

Then, naming these points: $p_1 = (p_{11}:p_{12}:p_{13}), p_2 = (p_{21}:p_{22}:p_{23}), p_3 = (p_{31}:p_{32}:p_{33})$ and $p_4 = (p_{41}:p_{42}:p_{43})$, we have that:

$$A = \begin{pmatrix} ap_{11} & bp_{21} & cp_{31} \\ ap_{12} & bp_{22} & cp_{32} \\ ap_{13} & bp_{23} & cp_{33} \end{pmatrix}$$

Satisfies $p_1 = \left[A\begin{pmatrix}1\\0\\0\end{pmatrix}\right], p_1 = \left[A\begin{pmatrix}0\\1\\0\end{pmatrix}\right], p_3 = \left[A\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$ and $p_4 = \left[A\begin{pmatrix}1\\1\\1\end{pmatrix}\right]$. Applying the same reasoning for $q_i's$

$$B = \begin{pmatrix} dq_{11} & eq_{21} & fq_{31} \\ dq_{12} & eq_{22} & fq_{32} \\ dq_{13} & eq_{23} & fq_{33} \end{pmatrix}$$

Satisfies $q_1 = \left[B\begin{pmatrix}1\\0\\0\end{pmatrix}\right], q_2 = \left[B\begin{pmatrix}0\\1\\0\end{pmatrix}\right], q_3 = \left[B\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$ and $q_4 = \left[B\begin{pmatrix}1\\1\\1\end{pmatrix}\right]$.

Since no three points are collinear, these two matrices are invertible. In fact, we only need the inverse of $A$ to be able to write the matrix we need as follows:

$$p_1 = \left[A\begin{pmatrix}1\\0\\0\end{pmatrix}\right] \text{ by definition 1 on page 67} \Longrightarrow p_1 = A\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right] \text{ multiplying by } A^{-1} \Longrightarrow A^{-1}[p_1] = A^{-1}A\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right] = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right]$$

Replace this final equation in $q_1 = \left[B\begin{pmatrix}1\\0\\0\end{pmatrix}\right]$ to obtain:

$$q_1 = \left[B\begin{pmatrix}1\\0\\0\end{pmatrix}\right] = B\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right] = BA^{-1}[p_1]$$

Showing that the matrix $BA^{-1}$ sends $p_1$ to $q_1$. A similar argument shows that this same matrix $BA^{-1}$ sends $p_2$ to $q_2$, $p_3$ to $q_3$ and $p_4$ to $q_4$. Finally, we know that the composition of linear transformations is a linear transformation and that the inverse of a linear transformation is a linear transformation. Therefore, $BA^{-1}$ is a linear transformation maps $p_i$ to $q_i$ for $i = 1,\dots,4$.

(Ex. 4) Let $(G,\circ,1)$ be a group. Define a relation $\sim$ on $G$ by $a \sim b$ if and only if there is an element $g \in G$ such that $b = g\circ a\circ g^{-1}$. Show that this relation is an equivalence relation. For $G = S_4$ the symmetric group of four elements, determine the partition $S_4$ into equivalence classes.

**Solution:** We need to show that the relation $\sim$ is Reflexive, Symmetric and Transitive.

Reflexivity: Suppose $a \in G$. Then $a = 1\circ a\circ 1^{-1} = 1\circ a\circ 1$. Hence, $a \sim a$.

Symmetry: Suppose $a,b \in G$ are such that $a \sim b$. Then, there exists $g \in G$ so that $b = g\circ a\circ g^{-1}$. Multiply this equation on the left by $g^{-1}$ and on the right by $g$ to get $g^{-1}\circ b\circ g = g^{-1}\circ g\circ a\circ g^{-1}\circ g = (g^{-1}\circ g)\circ a\circ(g^{-1}\circ g) = 1\circ a\circ 1 = a$, and hence $a = g^{-1}\circ b\circ g \iff b \sim a$.

Transitivity: Suppose $a,b,c \in G$ are such that $a \sim b$ and $b \sim c$. Then, there exists $g_1, g_2 \in G$ so that $b = g_1\circ a\circ g_1^{-1}$ and $c = g_2\circ b\circ g_2^{-1}$. Replace the first equation in the second:

$$\begin{aligned} c &= g_2\circ(g_1\circ a\circ g_1^{-1})\circ g_2^{-1} \\ &= (g_2\circ g_1)\circ a\circ(g_1^{-1}\circ g_2^{-1}) && \text{by associativity} \\ &= (g_2\circ g_1)\circ a\circ(g_2\circ g_1)^{-1} && \text{since } (g_2\circ g_1)^{-1} = g_1^{-1}\circ g_2^{-1} \end{aligned}$$

Hence, there exists $g = g_2\circ g_1$ such that $c = g_2\circ a\circ g_2^{-1} \iff a \sim c$.

To determine the partition of $S_4$ into equivalence classes under $\sim$, we will use two facts: (1) that we can write a permutation as the product of disjoint cycles and (2) that two permutations are conjugate if and only if they have the same cyclic structure (I take these as general facts and I won't prove them here). Let us begin with the class of the identity: $[()] = \{\sigma \in S_4 : \sigma \sim () \iff \text{there exists } \tau \in S_4 \text{ such that } \sigma = \tau()\tau^{-1} = ()\}$ , so the class of the identity is just the identity itself: $[()] = \{()\}$.
Next, $[(12)] = \{\sigma \in S_4 : \sigma \sim (12) \iff \text{there exists } \tau \in S_4 \text{ such that } \sigma = \tau(12)\tau^{-1} \iff \tau^{-1}\sigma\tau = (12)\}$.
Using fact (2), we get: $[(12)] = \{(12), (13), (14), (23), (24), (34)\}$. In a similar way we get:
$[(12)(34)] = \{(12)(34), (13)(24), (23)(14)\}$,
$[(123)] = \{(123), (132), (124), (142), (234), (243), (341), (314)\}, [(1234)] = \{(1234), (1243), (1324), (1342)(1423)(1432)\}$
Therefore, $G = [()] \cup [(12)] \cup [(12)(34)] \cup [(123)] \cup [(1234)]$

(Ex. 5) In the projective plane $\mathbb{F}_3P^2$, find a projective linear transformation in form of a $3 \times 3$ matrix that maps

$$(0:1:2) \mapsto (1:2:0) \mapsto (2:0:1) \mapsto (0:1:2)$$

**Solution:** Observe that we want to map points to permutations of said points. For instance $(0:1:2) \mapsto (1:2:0)$, has the effect of interchanging coordinates one with three, two with one and three with two. The same is true for all other mappings. So the linear projective linear transformation $A$ we want is the permutation matrix that has this effect, i.e.,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Clearly this is a linear transformation. This is a projective linear transformation since it is define up to nonzero multiples. In other words, the matrix $A$ and the matrix $2A$ would be the same. We can check that this matrix indeed does the job:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 2 \\ 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 2 \\ 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 2 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 2 + 1 \cdot 0 + 0 \cdot 1 \\ 0 \cdot 2 + 0 \cdot 0 + 1 \cdot 1 \\ 1 \cdot 2 + 0 \cdot 0 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$