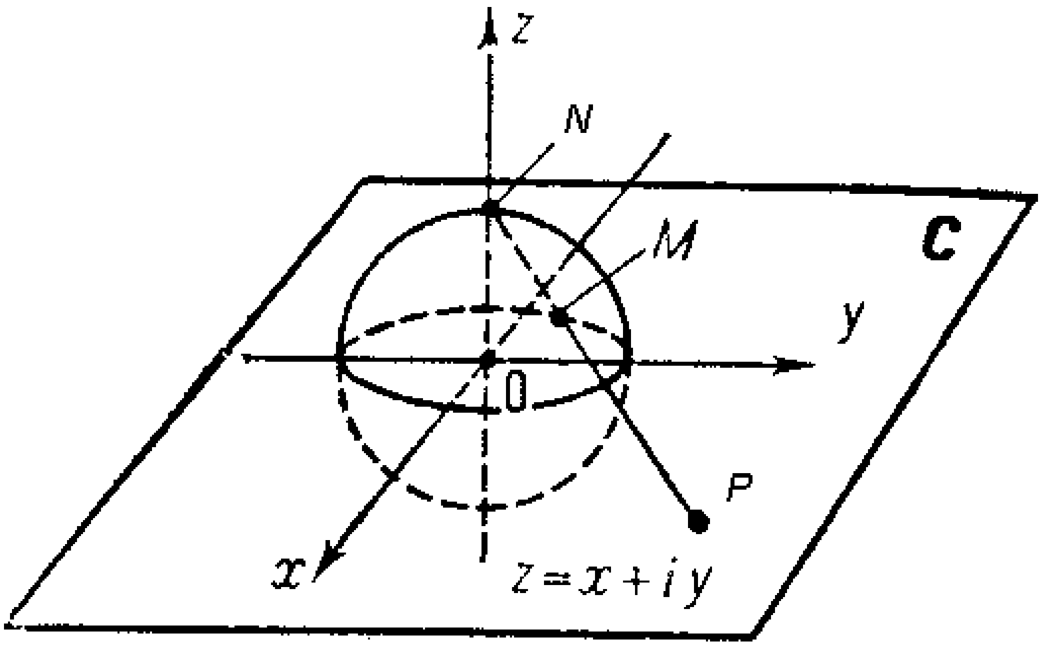
3.1 Stereographic Projection and the Riemann Sphere

Definition 52 *Let S^2 denote the unit sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$ and let $N=(0,0,1)$ denote the "north pole" of S^2. Given a point $M\in S^2$, other than N, then the line connecting N and M intersects the xy-plane at a point P. Then **stereographic projection** is the map*

$$\pi : S^2 - \{N\} \to \mathbb{C} : M \mapsto P.$$



Proposition 53 *The map π is given by*

$$\pi\left(a,b,c\right) = \frac{a+ib}{1-c}.$$

The inverse map is given by

$$\pi^{-1}\left(x+iy\right) = \frac{\left(2x,2y,x^2+y^2-1\right)}{1+x^2+y^2}.$$

Proof. Say $M=(a,b,c)$. Then the line connecting M and N can be written parametrically as

$$\mathbf{r}\left(t\right) = (0,0,1) + t\left(a,b,c-1\right).$$

This intersects the xy-plane when $1+t\left(z-1\right)=0$, i.e. when $t=\left(1-z\right)^{-1}$. Hence

$$P = \mathbf{r}\left(\frac{1}{1-c}\right) = \left(\frac{a}{1-c},\frac{b}{1-c}\right)$$

which is identified with

$$\frac{a+ib}{1-c} \in \mathbb{C}.$$

On the other hand, if $\pi(a,b,c) = x+iy$ then

$$\frac{a+ib}{1-c} = x+iy \quad \text{and} \quad a^2+b^2+c^2 = 1.$$

Hence $(a-ib)/(1-c) = x-iy$ and so

$$x^2+y^2 = \left(\frac{a+ib}{1-c}\right)\left(\frac{a-ib}{1-c}\right) = \frac{a^2+b^2}{(1-c)^2} = \frac{1-c^2}{(1-c)^2} = \frac{1+c}{1-c} = -1+\frac{2}{1-c},$$

giving

$$\frac{2}{1+x^2+y^2} = 1-c$$

and

$$c = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

Then

$$a+ib = \frac{2(x+iy)}{x^2+y^2+1}$$

and we may compare real and imaginary parts for the result. ■

Definition 54 *If we identify, via stereographic projection, points in the complex plane with points in $S^2 - \{N\}$ and further identify ∞ with N then we have a bijection between the extended complex plane $\tilde{\mathbb{C}}$ and S^2. Under this identification S^2 is known as the* ***Riemann sphere****.*

Corollary 55 *If M corresponds to $z \in \tilde{\mathbb{C}}$ then the antipodal point $-M$ corresponds to $-1/\bar{z}$.*

Proof. Say $M = (a,b,c)$ which corresponds to $z = (a+ib)/(1-c)$. Then $-M$ corresponds to

$$w = \frac{-a-ib}{1+c}.$$

Then

$$w\bar{z} = \frac{(a-ib)(-a-ib)}{(1-c)(1+c)} = \frac{-a^2-b^2}{1-c^2} = \frac{c^2-1}{1-c^2} = -1.$$

■

Theorem 56 *Circlines in the complex plane correspond to circles on the Riemann sphere and vice-versa.*

Proof. Consider the plane Π with equation $Aa+Bb+Cc = D$. This plane will intersect with S^2 in a circle if $A^2+B^2+C^2 > D^2$. Recall that the point corresponding to $z = x+iy$ is

$$(a,b,c) = \frac{(2x, 2y, x^2+y^2-1)}{1+x^2+y^2}$$

which lies in the plane $Aa+Bb+Cc = D$ if and only if

$$2Ax+2By+C\left(x^2+y^2-1\right) = D\left(1+x^2+y^2\right).$$

This can be rewritten as

$$(C-D)\left(x^2+y^2\right)+2Ax+2By+(-C-D)=0.$$

This is the equation of a circle in $\mathbb{C}$ if $C \neq D$. The centre is $(A/(D-C), B/(D-C))$ and the radius is

$$\frac{\sqrt{A^2+B^2+C^2-D^2}}{C-D}.$$

Furthermore all circles can be written in this form — we can see this by setting $C-D=1$ and letting $A, B, C+D$ vary arbitrarily. On the other hand if $C=D$ then we have the equation

$$Ax+By=C$$

which is the equation of a line — and moreover any line can be written in this form. Note that $C=D$ if and only if $N=(0,0,1)$ lies in the plane. So under stereographic projection lines in the complex plane correspond to circles on S^2 which pass through the north pole. ■

Corollary 57 *The great circles on S^2 correspond to circlines of the form*

$$\alpha(z\bar{z}-1)+\bar{\beta}z+\beta\bar{z}=0.$$

Proof. The plane Π makes a great circle on S^2 if and only if the plane contains the origin — i.e. if and only if $D=0$. The corresponding $x+iy$ satisfy the equation

$$2Ax+2By+C\left(x^2+y^2-1\right)=0$$

If we set $\alpha=C$ and $\beta=A+iB$ then the result follows. ■

Proposition 58 *Stereographic projection is conformal (i.e. angle-preserving).*

Proof. Without loss of generality we can consider the angle defined by the real axis and an arbitray line meeting it at the point $p \in \mathbb{R}$ and making an angle θ. So points on the two lines can be parametrised as

$$z=p+t, \qquad z=p+te^{i\theta},$$

where t is real. These points map onto the sphere as

$$\mathbf{r}(t)=\frac{\left(2(p+t), 0, (p+t)^2-1\right)}{1+(p+t)^2}, \qquad \mathbf{s}(t)=\frac{\left(2(p+t\cos\theta), 2t\sin\theta, (p+t\cos\theta)^2+t^2\sin^2\theta-1\right)}{1+(p+t\cos\theta)^2+t^2\sin^2\theta}.$$

Then

$$\mathbf{r}'(0)=\frac{\left(2\left(p^2-1\right), 0, 4p\right)}{\left(1+p^2\right)^2}, \qquad \mathbf{s}'(0)=\frac{\left(2\left(p^2-1\right)\cos\theta, 2\left(1+p^2\right)\sin\theta, 4p\cos\theta\right)}{\left(p^2+1\right)^2}.$$

So the angle ϕ between these tangent vector is given by

$$\begin{aligned}
\cos\phi &= \frac{\left(4\left(p^2-1\right)^2\cos\theta+0+16p^2\cos\theta\right)}{\sqrt{4\left(p^2-1\right)^2+16p^2}\sqrt{4\left(p^2-1\right)^2\cos^2\theta+4\left(1+p^2\right)^2\sin^2\theta+16p^2\cos^2\theta}} \\
&= \frac{4\left(p^2+1\right)^2\cos\theta}{\left\{2\left(p^2+1\right)\right\}\left\{2\left(p^2+1\right)\right\}} \\
&= \cos\theta.
\end{aligned}$$

Hence stereographic projection is conformal as required. ■