Markov Chains

M447 - Mathematical Models/Applications 1
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Let us start by discussing an *example*.
Consider the Markov Chain with 4 states whose transition probability matrix is given by:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{cccc} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/3 & 7/10 & 0 \\ 0 & 2/10 & 8/10 & 0 \\ 1/10 & 0 & 0 & 9/10 \end{array} \right\| \end{array}$$

If we try to find for the long-term fraction of time spend in each state by solving $wP = w$ directly, i.e., finding the left eigenvector with eigenvalue 1, we will get (using mathematica):

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Eigenvectors@Transpose[P]
        {{1/5, 0, 0, 1}, {0, 2/7, 1, 0}, {-1, 0, 0, 1}, {0, -1, 1, 0}}
Eigenvalues@Transpose[P]
        {1, 1, 2/5, 1/10}
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First note that eigenvectors that change signs cannot possibly be normalized to provide a probability distribution so ignore these. We can see that there are two possible left eigenvectors with eigenvalue 1 specifically $(1/5 \quad 0 \quad 0 \quad 1)$ and $(0 \quad 2/7 \quad 1 \quad 0)$. So, in this case there is no certainty as to what is the long-term fraction of time spend in each state since it actually depends on where you start the chain.

Looking back at the definition of $\mathbf{P}$, this example suggests that a disconnected (or non-ergodic) Markov Chain has no unique long-term distribution of time spend in each state. Let us try to prove that an ergodic Markov Chain has a unique long-term distribution and that the stable vector does not change sign.

In what follows suppose that $\mathbf{P}$ is the transition matrix of an ergodic Markov Chain.
Theorem 1: If w is a real solution to $w = w\mathbf{P}$, then w does not take different signs.

Proof : (by Contradiction). Suppose that the elements of w take different signs, i.e., $w = (w_1 \quad w_2 \quad \cdots \quad w_n)$.

Define $u' = \begin{pmatrix} sign(w_1) \\ sign(w_2) \\ \vdots \\ sign(w_n) \end{pmatrix}$ where, $sign(w_i) = \begin{cases} 1 & \text{if } w_i \geq 0 \\ \text{-1} & \text{if } w_i < 0 \end{cases}$

Recall that the product of matrices is associative and hence, $(w\mathbf{P})u' = w(\mathbf{P}u')$. Using this fact

$$\begin{aligned} w\mathbf{P}u' &= (w\mathbf{P})u' \\ &= wu' \qquad \text{By hypothesis } w = w\mathbf{P} \\ &= \sum_{i=1}^{n} |w_i| \end{aligned}$$

$$\begin{aligned} w\mathbf{P}u' &= w(\mathbf{P}u') \\ &= w \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{where } |x_i| \leq 1 \end{aligned}$$

If, for some i is true that $|x_i| < 1$, then $|w\mathbf{P}u'| = |(w\mathbf{P})u'| > |u(\mathbf{P}u)| = |w\mathbf{P}u'|$, so it follows $|w\mathbf{P}u'| > |w\mathbf{P}u'|$ a contradiction.

Therefore, for all i we must have $|x_i| = 1$ and $sign(x_i) = sign(w_i)$.$\square$

Theorem 2: The stable vector of the chain $\mathbf{P}$ is unique.

Proof : Let $u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Then $\mathbf{P}u = u$, because the rows of $\mathbf{P}$ add up to 1.

Suppose λ is an eigenvalue of the matrix $\mathbf{P}$ with associated eigenvector w. Consider the following:

$$\begin{aligned} &w\mathbf{P} = \lambda w && \text{assumption} \\ &w\mathbf{P}u = \lambda wu && \text{multiply both sides by } u \\ &w(\mathbf{P}u) = \lambda(wu) && \text{associativity} \\ &wu = \lambda(wu) && \text{since } \mathbf{P}u = u \\ &\implies \lambda = 1 && \end{aligned}$$

So the stable vector is the only vector with eigenvalue $\lambda = 1$. $\square$

Remark: Together *Theorem 1* and *Theorem 2* prove that if a Markov Chain is connected, then there is a unique solution to $w = w\mathbf{P}$ and furthermore, the vector w can be written as a probability vector.