M451/551 Exam 1

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INSTRUCTIONS: Please make sure your exam has 7 pages, in addition to this cover page. **You must justify your solutions to receive credit.** Please try to fit your solutions into the space provided. If you do need extra space, please write "continued on the back," and continue on the back of the same sheet. Also, be sure to indicate your final answer to each problem clearly.

Do not write below this line. For graders use:

1.	10	5.	12
2.	11	6.	15
3.	9	7.	8
4.	10	Sum	75

Formulae:

Black-Scholes:

$$C(S,K,T,r,\sigma) = S\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}) \quad \text{where} \quad \omega = \frac{(r+\sigma^2/2)T - \log\frac{K}{S}}{\sigma\sqrt{T}}$$

PDF of Standard Normal Random Variable $Z_{0,1}$:

$$\Phi'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Put-Call Parity:

$$S + P - C = Ke^{-rT}$$

$3+5+7=15$

Problem 1. (10 PTS.) *A bag holds 3 red marbles 5 blue marbles and 7 green marbles. One marble is chosen from the bag at random. Suppose X represents the number of marbles with the same color as the chosen marble (including the one chosen).*

(a) *Calculate* $E[X]$.

(b) *What is the conditional probability that* $X > 4$ *given that the chosen marble is either red or green?*

X = # of marbles with same color as chosen marble.

$X \in \{3, 5, 7\}$ depending on choosen marble.

Following table summarizes X:

$X=x$	$P(X=x)$	$x \cdot P(X=x)$
3	3/15	9/15
5	5/15	25/15
7	7/15	49/15

(a)

$$E[x] = \frac{9+25+49}{15} = \frac{34+49}{15} = \boxed{\frac{83}{15}}$$

(b) $P(X>4 \mid \text{marble is R or G}) =$

$$\frac{P(X>4 \text{ and } (R \text{ or } G))}{P(R \text{ or } G)} = \frac{P((X>4 \text{ and } R) \text{ or } (X>4 \text{ and } G))}{10/15}$$

$X>4$ and $R = \phi$ (no way of this happening.

$X>4$ and $G = G$, hence

$$\frac{P(G)}{10/15} = \frac{7/15}{10/15} = \boxed{\frac{7}{10}}$$

+10

Problem 2. (15 Pts.) *Consider two yearly income streams in dollars where the first payment is made immediately:*

$$A: \ 100, \ 80, \ 200$$
$$B: \ 90, \ 100, \ 195$$

Assuming simple annual compounding, for what nominal interest rates $r \in (-1,1)$ do the present values satisfy $PV(A) > PV(B)$?

Let $\alpha = (1+r)^{-1}$, then

$PV(A) = 100 + 80\alpha + 200\alpha^2$ AND $PV(B) = 90 + 100\alpha + 195\alpha^2$.

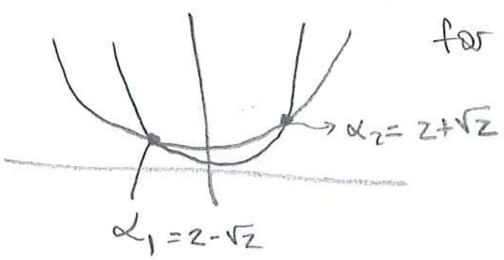
$\Rightarrow PV(A) = PV(B) \Rightarrow 100 + 80\alpha + 200\alpha^2 = 90 + 100\alpha + 195\alpha^2$

$\Rightarrow 5\alpha^2 - 20\alpha + 10 = 0 \Rightarrow \alpha = \dfrac{20 \pm \sqrt{400-200}}{10} = \dfrac{20 \pm \sqrt{200}}{10} = \dfrac{20 \pm 10\sqrt{2}}{10}$

So, $\alpha = 2 \pm \sqrt{2}$,

Since both of this are quadratic functions, we have a situation like:

for polynomials on α



$\Rightarrow$ the domain gets partitioned into 3 intervals: $(-\infty, \alpha_1) \cup [\alpha_1, \alpha_2), \cup [\alpha_2, \infty)$

Hence, it suffices to check one of these to determine what happens every where. Let's check $1 \in (\alpha_1, \alpha_2)$:

For $\alpha = 1$ we have $1 = (1+r)^{-1} \Rightarrow 1 = 1+r \Rightarrow \underline{r = 0}$

$PV(A) = 100 + 80 + 200 = 380 > 285 = 90 + 100 + 195 = PV(B)$

Hence, since $0 \in (-1,1)$ $PV(A) > PV(B)$, it follows that for all interest rates $r \in (-1,1)$, $PV(A) > PV(B)$

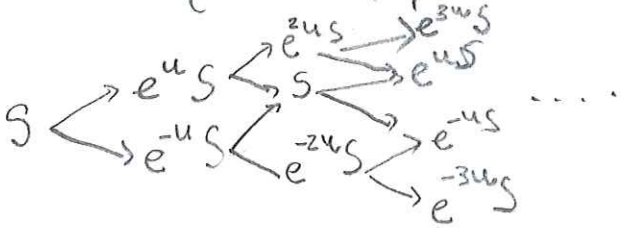
+11

Problem 3. (15 PTS.) *A model for the movement of a stock supposes that, if the present price of the stock is S, then after one time period it will either be $e^u * S$ with probability p or $e^{-u} * S$ with probability $1-p$. (Here $u > 1$ and $1/4 < p < 3/4$.)*

(a) *Assuming that successive movements are independent, write an expression involving Φ, p and u which approximates the probability that the stock's price will double after the next 1,000 time periods. Hint: if X_i is a random variable that takes the value $+1$ with probability p and -1 with probability $(1-p)$ then $E[X_i] = 2p-1$ and $Var(X_i) = 4p(1-p)$.*

(b) *Very briefly explain why your approximation is valid.*

(a) $X_i = \begin{cases} 1 & \text{with prob } p \\ -1 & \text{with prob } (1-p) \end{cases} \Rightarrow E[X_i] = 2p-1; \ Var(X_i) = 4p(1-p)$



The stock price at the n^{th} level is given by:

$S(n) \cancel{\sum_{i=1}^{n} \binom{n}{k}} e^{Y}S \, e^{n-Y}S$, where $Y = \sum X_i \sim Bin(n, 2p-1)$

By Central limit theorem, the sum of i.i.d r.v. converges in probability to a normal R.V. with mean $nE[X_i]$ X

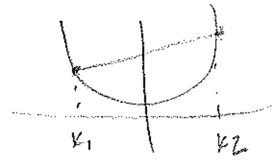
So, $P(S(n) > 2S) \overset{\text{why?}}{\approx} P(X_{n(2p-1), n4p(1-p)} > 2S)$

$= P(X_{0,1} > (2S - n(2p-1))/\sqrt{n4p(1-p)})$ +6

$= 1 - \Phi\left(\frac{2S - n(2p-1)}{\sqrt{n4p(1-p)}}\right)$ ... Normalizing mean 0 stdev 1

(b) Like I mentioned before, my approximation is valid because as we sum more X_i, the movement of the stock is well approximated as a normal r.v. X
I belive 1,000 movements is a number big enough to produce a good approximation.

+3



Problem 4. (15 Pts.) *Let $P(K,T)$ denote the cost of a European put option with strike K and expiration time T. Prove that $P(K,T)$ is convex in K for fixed T. (If you rely on the convexity of other functions, then prove their convexity first.)*

To prove that $P(K,T)$ is convex, we use that $C(K,T)$ is convex and then use put-call parity formula to derive convexity for $P(K,T)$.

that $C(K,T)$ is convex means:

$$\forall K_1, K_2 \text{ AND } \lambda \in (0,1): \quad \lambda C(K_1,T) + (1-\lambda) C(K_2,T) \geq C(\lambda K_1 + (1-\lambda) K_2, T).$$

We want to show:

$$\forall K_1, K_2 \text{ AND } \lambda \in (0,1) \quad \lambda P(K_1,T) + (1-\lambda) P(K_2,T) \geq P(\lambda K_1 + (1-\lambda) K_2, T).$$

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But by Put-Call parity

$$S + P(K,T) - C(K,T) = Ke^{-rT} \Rightarrow P(K,T) = Ke^{-rT} - S + C(K,T).$$

Hence

$$\lambda P(K_1,T) + (1-\lambda) P(K_2,T) = \lambda(Ke^{-rT} - S + C(K_1,T)) + (1-\lambda)(Ke^{-rT} - S + C(K_2,T))$$

$$= \cancel{\lambda Ke^{-rT}} - \cancel{\lambda S} + \lambda C(K_1,T) + Ke^{-rT} - S + C(K_2,T) - \cancel{\lambda Ke^{-rT}} - \cancel{\lambda S} - \lambda C(K_2,T)$$

$$= Ke^{-rT} - S + \lambda C(K_1,T) + (1-\lambda) C(K_2,T) \quad \ldots\ldots \text{By hyp.}$$

$$\geq Ke^{-rT} - S + C(\lambda K_1 + (1-\lambda) K_2, T)$$

$\Rightarrow$ Again by put-call

$$\lambda P(K_1,T) + (1-\lambda) P(K_2,T) \geq \cancel{Ke^{-rT}} - \cancel{S} + \cancel{S} - \cancel{Ke^{-rT}} + P(\lambda K_1 + (1-\lambda) K_2, T)$$

Proving convexity.

That $C(K,T)$ is convex can be shown by taking the second derivative of B-S and concluding it is always positive

Yes

Problem 5. (15 Pts.) *An experiment can result in any of the outcomes 1, 2, or 3. Suppose the profit matrix for two different wagers is given by*

$$r_1(1) = 1, \qquad r_1(2) = 2, \qquad r_1(3) = -3,$$
$$r_2(1) = 3, \qquad r_2(2) = 3, \qquad r_2(3) = -5.$$

Decide if there is an arbitrage opportunity in which case come up with a positive betting strategy, or else find the risk neutral probabilities p_1, p_2, p_3.

According to the Arbitrage theorem, there will be arbitrage if we can't solve for:

$R \cdot p = 0$, where $R = \begin{bmatrix} r_1(1) & r_1(2) & r_1(3) \\ r_2(1) & r_2(2) & r_2(3) \end{bmatrix}$ and $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$,

where $p_i \geq 0$ and $\sum p_i = 1$. let us try to solve this:

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 3 & -5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} p_1 + 2p_2 - 3p_3 = 0 \\ p_1 + 3p_2 - 5p_3 = 0 \\ p_1 + p_2 + p_3 = 1 \end{matrix}$$ subtract :

$$4p_3 = 1 \Rightarrow \boxed{p_3 = \frac{1}{4}} \Rightarrow \begin{cases} p_1 + p_2 = \frac{3}{4} \\ p_1 + 3p_2 = \frac{5}{4} \\ p_1 + p_2 = \frac{3}{4} \end{cases} \Rightarrow \begin{cases} p_1 = \frac{3}{4} - p_2 \\ \frac{3}{4} - p_2 + 3p_2 = \frac{5}{4} \end{cases} \Rightarrow 2p_2 = \frac{1}{2} \Rightarrow \boxed{p_2 = \frac{1}{4}}$$

And so, $p_1 + p_2 + p_3 = 1 \Rightarrow p_1 + \frac{1}{4} + \frac{1}{4} = 1 \Rightarrow \boxed{p_1 = \frac{1}{2}}$

The risk neutral probabilities are: $\boxed{(p_1, p_2, p_3) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)}$

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Problem 6. (15 PTS.) *Assuming continuous compounding and a nominal yearly interest rate of r, the price of a security follows a risk-neutral geometric Brownian motion with volatility parameter σ and drift parameter $r - \frac{\sigma^2}{2}$. The current price of the security is $S(0)$. A European cash-or-nothing option pays its holder a fixed amount F if the price $S(T)$ at the expiration time T is larger than K and pays 0 otherwise. Find the no-arbitrage price $O(S(0), K, F, T, r, \sigma)$ for this option. (Your expression may also involve the Φ function.)*

$O(S(0), K, F, T, r, \sigma)$ is the probability that the option will be exercised times the payoff, discounted to Present Value, i.e.,

$$O(S(0), K, F, T, r, \sigma) = P(S(T) > K) F e^{-rT}, \quad \text{where}$$

$$P(S(T) > K) = P\left(\frac{S(T)}{S(0)} > \frac{K}{S(0)}\right), \quad \text{here } \frac{S(T)}{S(0)} \text{ follows G.M.B.}$$

$$= P\left(X > \log\left(\frac{K}{S(0)}\right)\right), \quad X \sim \text{Normal}\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

$$= P\left(\frac{X-u}{\sigma} > \left(\log\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T\right) / \sigma\sqrt{T}\right)$$

$$= 1 - \Phi\left(\left(\log\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T\right) / \sigma\sqrt{T}\right), \quad \Phi \text{ is the normal cdf.}$$

So, the price is given by:

$$\boxed{O(S(0), K, F, T, r, \sigma) = \left(1 - \Phi\left(\left(\log\left(\frac{K}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)T\right) / \sigma\sqrt{T}\right)\right) F e^{-rT}}$$

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Problem 7. (15 PTS.) *What should the price $P(S, K, T, r, \sigma)$ of a European put option become in the limit as the volatility σ tends to 0? (Hint: first compute it for the corresponding call option.)*

First I'll use B-S for the corresponding call and then use the Put-Call parity formula to determine the put price.

B-S: $C(S,K,T,r,\sigma) = S\,\Phi(w) - Ke^{-rT}\,\Phi(w-\sigma\sqrt{T})$,

where $w = \dfrac{(r+\sigma^2/2)T - \log\frac{K}{S}}{\sigma\sqrt{T}}$

Take $\lim_{\sigma\to 0} w = \lim_{\sigma\to 0} \dfrac{rT}{\sigma\sqrt{T}} + \dfrac{(\sigma^2/2)T}{\sigma\sqrt{T}} - \dfrac{\log K/S}{\sigma\sqrt{T}} = \infty + 0 - \infty$ indet.

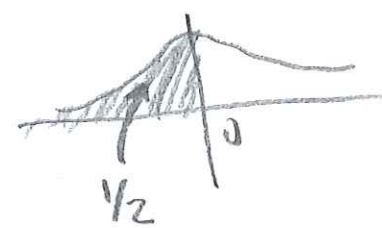
USE L'Hopital: $\lim_{\sigma\to 0} \dfrac{rT + \sigma^2/2 - \log(\frac{K}{S})}{\sigma\sqrt{T}} = \lim_{\sigma\to 0} \dfrac{(rT+\sigma^2/2 - \log(K/S))'}{(\sigma\sqrt{T})'}$, (If it exists)

only if num $\to 0$

$= \lim_{\sigma\to 0} \dfrac{\sigma}{\sqrt{T}} = 0$. So $w \to 0$ as $\sigma \to 0$.

$\lim_{\sigma\to 0} C(S,K,T,r,\sigma) = S\,\Phi(0) - Ke^{-rT}\,\Phi(0)$

$= \frac{1}{2}(S - Ke^{-rT})$, since Φ is the normal cdf



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