Math Finance 3-3-15

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Recall we had B.M. $W(t)$ with drift μ & vol. σ
($W(t) - W(s)$ was normal R.V. with mean $\mu(t-s)$ var. $\sigma^2(t-s)$)

Now we eneralize to a process $\{X_t\}$ where both μ and σ can vary with t. Suppose:

$X_t - X_s$ has mean $= \int_s^t \mu(r)\,dr$ and

$$\text{var} = \int_s^t (\sigma(r))^2\,dr.$$

(original case was $\mu(r) \equiv \mu$ $\sigma(r) \equiv \sigma$.)

recall $X_{t+h} - X_t \xrightarrow{h \to 0} 0$ but not as fast as h so $\dfrac{X_{t+h} - X_t}{h}$ blows up

so not diff. <u>but</u>

$$X_{t+h} - X_t = \int_t^{t+h} \mu(r)\,dr + \left(\int_t^{t+h} \sigma(r)^2\,dr\right)^{1/2} Z_{0,1}$$

$$= h\mu(t) + \int_t^{t+h} (\mu(r) - \mu(t))\,dr + \sqrt{h}\,\sigma(t)\,Z_{0,1}$$

$$\underbrace{\left(\left(h\sigma^2(t) + \int_t^{t+h} (\sigma^2(r) - \sigma^2(t))\,dr\right)^{1/2} - \sqrt{h}\,\sigma(t)\right)}_{g(t,h)} Z_{0,1}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{g(t,h)}$$

Now $g(t,h) = \sqrt{h}\,\sigma(t)\left(\left(1+\frac{1}{h}\int_t^{t+h}\left(\frac{\sigma^2(r)}{\sigma^2(t)}-1\right)dr\right)^{1/2}-1\right)$

and suppose $|\sigma^2(r)-\sigma^2(t)| \leq (r-t)^\alpha C\sigma^2(t)$

then

$$g(t,h) \leq \sqrt{h}\,\sigma(t)\left(\left(1+\frac{C}{h}\int_t^{t+h}(r-t)^\alpha dr\right)^{1/2}-1\right)$$

$$= \sqrt{h}\,\sigma(t)\left(\left(1+\frac{C\,h^{\alpha+\cancel{1}}}{\cancel{h}(\alpha+1)}\right)^{1/2}-1\right)$$

$$\approx \sqrt{h}\,\sigma(t)\,\frac{C\,h^\alpha}{2\alpha}.$$ So for $\alpha > 1/2$

this converges to 0 as $h \to 0$ faster than h

similarly $\int_t^{t+h} \mu(r)-\mu(t)\,dr \xrightarrow{h\to 0} 0$ faster than h

if $\mu(t)$ continuous. So if $\mu \in C^0$ & $\sigma^2 \in C^{1/2+\varepsilon}$.

We have: $(*)$

$X_{t+h} - X_t - h\mu(t) - \sqrt{h}\,\sigma(t)\,Z_{0,1}$ is $o(h)$

Def If X_t satisfies $*$ then we say

X_t satisfies the stochastic differential equation

$$dX = \mu(t)\,dt + \sigma(t)\,dW_t$$

(here W_t is the B.M. with drift 0 & vol $\sigma = 1$)

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Such a stochastic process is called an <u>Itô process</u>

<u>Ex</u> If $\sigma(t) \equiv 0$ then $dX_t = \mu(t)\,dt$

means $\lim_{h\to 0} \dfrac{X_{t+h} - X_t - h\mu(t)}{h} = 0$

or $\lim_{h\to 0} \dfrac{X_{t+h} - X_t}{h} = \mu(t)$

or $\dfrac{dX_t}{dt} = \mu(t)$

(usual derivative exists)

If we have a function $f(x,t) \in C^1$ and X_t is an Itô process with $\sigma \equiv 0$ (so X_t also differentiable) then we can write usual multivariate chain-rule

as: $\dfrac{d}{dt}\left(f(X_t,t)\right) = \dfrac{\partial f}{\partial x}(X_t,t)\,\mu(t) + \dfrac{\partial f}{\partial t}(X_t,t)$

Q: But what about when $\sigma(t) \not\equiv 0$?

Answer:

<u>Thm</u> (Itô's Lemma) Let X_t be an Ito process with

$$dX_t = \mu(t)\,dt + \sigma(t)\,dW_t \quad \text{and}$$

$f(x,t)$ a C^2 function. Then $f(X_t,t)$ is an Itô process with

$$d(f(X_t,t)) = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}(X_t,t)\,dX_t + \frac{\partial^2 f}{\partial x^2}(X_t,t)\,\frac{\sigma^2(t)}{2}\,dt$$

$$= \left(\frac{\partial f}{\partial t}(X_t,t) + \frac{\partial f}{\partial x}(X_t,t)\mu(t) + \frac{\partial^2 f}{\partial x^2}(X_t,t)\frac{\sigma^2(t)}{2}\right)dt + \frac{\partial f}{\partial x}\sigma(t)\,dW_t$$

Ex $S_t = S_0 e^{X_t}$ then $f(x,t) = S_0 e^{x}$

$$d(S_t) = \left(0 + \mu(t) S_0 e^{X_t} + \frac{\sigma^2(t)}{2} S_0 e^{X_t}\right)dt + \sigma(t) S_0 e^{X_t} dW_t$$

$$= \left(\mu(t) + \frac{\sigma^2(t)}{2}\right) S_t \, dt + \sigma(t) S_t \, dW_t$$

Pf (of Thm) By f being twice diff. we have:

$$f(X_{t+h}, t+h) - f(X_t, t) = \frac{\partial f}{\partial t}(X_t, t) \cdot h +$$

$$\frac{\partial f}{\partial x}(X_t, t)(X_{t+h} - X_t) + \frac{\partial^2 f}{\partial x^2}(X_t, t) \frac{(X_{t+h} - X_t)^2}{2}$$

$$+ O\left((X_{t+h} - X_t)^3, h^2\right) \quad \text{(Taylor Series)}$$

but $X_{t+h} - X_t$ is $\mu(t) h + \sqrt{h}\, \sigma(t) Z_{0,1}$

so $(X_{t+h} - X_t)^2 = \mu^2 h^2 + 2\mu(t)\sigma(t) h^{3/2} Z_{0,1} + h\sigma^2 Z_{0,1}^2$

Now $Z_{0,1}^2$ still has mean $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx = 1$ ← odd

On the other hand $\mathrm{Var}(h\sigma^2 Z_{0,1}^2) = h^2 \sigma^4 \cdot \mathrm{Var}(Z_{0,1}^2)$ 8

and so it is $o(h)$ – i.e. it goes like h^{β} for $\beta > 1$. Similarly all other moments are $o(h)$ also. Since $\mu(t)^2 h^2$ and $2\mu(t)\sigma(t) h^{3/2}$ are $o(h)$ the only term that is not is the mean $h\sigma(t)^2$.

This leaves:

$$f(X_{t+h}, t+h) - f(X_t, t) = \frac{\partial f}{\partial t}(x_t, t)\, h +$$

$$\frac{\partial f}{\partial x}(x_t, t)\, \mu(t) h + \frac{\partial f}{\partial x}(x_t, t) \sqrt{h}\, \sigma(t) Z_{0,1} + \frac{\partial^2 f}{\partial x^2}(x_t, t) \frac{h\sigma^2}{2}$$

$$+ o(h)$$

Now $Z_{0,1} = \frac{1}{\sqrt{h}} (W_{t+h} - W_t)$ so we get

$$f(X_{t+h}, t+h) - f(X_t, t) = \frac{\partial f}{\partial t} h + \frac{\partial f}{\partial x}\left(\mu(t) h + \sigma(t)(W_{t+h} - W_t)\right)$$

$$+ \frac{\partial^2 f}{\partial x^2} h \frac{\sigma^2}{2}$$ letting $h \to dt$

$$df(X_t, t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu(t)\right) dt + \frac{\partial f}{\partial x}\sigma(t) dW_t + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} dt$$

$\blacksquare$

We can now derive Black-Scholes eqn. If $f(x,t) = C(S_0 e^x, t)$, where here t is the ~~time~~ the option is purchased, not the expiry T, then $f(X_t, t) = C(S_0 e^{X_t}, t)$

$= C(S_t, t)$ If you apply Itô's Lemma then in your homework you will show that (after rewriting everything in terms of $S_t = S_0 e^{X_t}$) you obtain:

$$dC = \frac{dC}{dt} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt$$

by Itô.

Consider a portfolio of the option & α stocks.

$$d(C+\alpha S) = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \alpha \mu S \right) dt$$

$$+ \left(\sigma S \frac{\partial C}{\partial S} + \alpha \sigma S \right) dW_t$$

set $\alpha = -\frac{\partial C}{\partial S}$ (Δ-hedge) then

the coefficient of the dW_t term becomes zero so this portfolio is riskless and completely deterministic. Therefore its drift must be $r(C+\alpha S)$ or in other words over time it's value is $e^{rt}(C+\alpha S)$

so $\frac{d}{dt}(C+\alpha S) = r(C+\alpha S)$ and so

gives: $$r\left(C - S\frac{\partial C}{\partial S}\right) = \frac{\partial C}{\partial t} + \cancel{\mu S \frac{\partial C}{\partial S}} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} - \cancel{S\mu \frac{\partial C}{\partial S}}$$

or $=$ $$\boxed{\frac{\partial C}{\partial t} = rC - rS\frac{\partial C}{\partial S} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2}}$$

This is an ordinary PDE which can be solved directly.

(Remember here $S = S_t$ $\sigma = \sigma(t)$ and $r = r(t)$ may depend on t, and even S_t)

This $C(S_t, t)$ is the call price at time t

so in terms of our former naming of the call price @ time 0 which we called $C(S_0, k, T, r, \sigma)$ we have

$$C(S_t, t) = C(S_t, K, T-t, r, \sigma)$$

but we only derived the formula for this when σ is constant.