M463 Homework 17

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July 19, 2013

4.4#10c Let Z be a standard normal random variable. Find a formula for the density of $1/Z$.

Solution: In this case $g(Z) = 1/Z$, which means that $g'(Z) = -1/Z^2$. Applying the formula:

$$f_y(y) = \sum_{\{z:g(z)=y\}} \frac{f_Z(z)}{|g'(z)|} = \sum_{\{z=1/y\}} \frac{f_Z(z)}{|g'(z)|} = \left.\frac{\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}}{\left|-\frac{1}{z^2}\right|}\right|_{z=1/y} = \left.\frac{z^2 e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right|_{z=1/y} = \boxed{\frac{e^{-\frac{1}{2y^2}}}{\sqrt{2\pi}y^2}}, \quad \text{for } -\infty < y < 0; 0 < y < \infty$$

4.5#8c Components in the following series-parallel systems have independent exponentially distributed lifetimes. Component i has mean lifetime μ_i. Find a formula for the probability that the system operates for at least t units of time, and sketch the graph of this function of t in case $\mu_i = i$ for each i.

Solution: Let T_1 = lifetime of component i, for $i = 1, 2, 3, 4$. By hypothesis, $T_i \sim Exp\left(\lambda_i = \frac{1}{\mu_i}\right)$.

Note that the p.d.fs are $F_i(t) = 1 - e^{-t/\mu_i}$ and the survival functions are $S_i(t) = 1 - F_i(t) = e^{-t/\mu_i}$.

Now, let T = lifetime of the system. By the diagram we get that $T = max(min(T_1, T_2), min(T_3, T_4))$.

We want to compute the survival function for T, i.e., the following probability:

$$P(T > t) = 1 - P(T \leq t)$$

Let $Z = min(T_1, T_2)$ and $W = min(T_3, T_4)$. Then we can compute the c.d.f of T as follow:

$$P(T \leq t) = P(Z \leq t, W \leq t) = P(Z \leq t)P(W \leq t) = F_Z(t)F_W(t)$$

where $F_Z(t) = P(Z \leq t) = 1 - P(Z > t) = 1 - P(T_1 > t, T_2 > t) = 1 - [P(T_1 > t)P(T_2 > t)] = 1 - e^{-t(1/\mu_1 + 1/\mu_2)}$

Likewise, $F_W(t) = 1 - e^{-t(1/\mu_3 + 1/\mu_4)}$. Finally, we can combine all these expressions:

$$P(T > t) = 1 - P(T \leq t) = \boxed{1 - [(1 - e^{-t(1/\mu_1 + 1/\mu_2)})(1 - e^{-t(1/\mu_3 + 1/\mu_4)})]}$$

In case $\mu_i = i$ for each i we get the function:

$$S_T(t) = 1 - [(1 - e^{-t(1+1/2)})(1 - e^{-t(1/3+1/4)})] = 1 - [(1 - e^{-3t/2})(1 - e^{-7t/12})]$$

The following is the graph of this function: