M463 Homework 20

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Let X and Y be independent standard normal variables.

a) For a constant k, find $P(X > kY)$.

Solution:

$$P(X > kY) = P(X - kY > 0) = 1 - P(X - kY \leq 0)$$

If we let $V = X - kY$, then $V \sim Normal\left(\mu_V = 0,\ \sigma_V = \sqrt{1+k^2}\right)$, since the sum of independent normal R.V.s is normal. Moreover, we can compute the mean and variance as follow:

$$\mu_V = E(V) = E(X - kY) = E(X) - kE(Y) = 0 - k \cdot 0 = 0$$

$$\sigma_V^2 = Var(X - kY) = \text{by independence} = Var(X) + (-k)^2 Var(Y) = 1 + k^2 \Longrightarrow \sigma_V = \sqrt{1+k^2}$$

Now we can compute the following probability:

$$1 - P(V \leq 0) = 1 - P(V* \leq \frac{0-0}{\sqrt{1+k^2}}) = 1 - P(V* \leq 0) = 1 - \Phi(0) = \boxed{0.5}$$

b) If $U = \sqrt{3}X + Y$, and $V = X - \sqrt{3}Y$, find $P(U > kV)$.

Solution: First note that both U and V are sums of independent normal R.V.s and thus, they are normal. Their parameters are:

$$U, V \sim Normal(\mu = 0, \sigma = 2), \qquad \text{since:}$$

$$\mu = E(U) = E(\sqrt{3}X + Y) = \sqrt{3}E(X) + E(Y) = \sqrt{3} \cdot 0 + 0 = 0 = E(V)$$

$$\sigma^2 = Var(U) = Var(\sqrt{3}X + Y) = \text{by independence} = (\sqrt{3})^2 Var(X) + Var(Y) = 3 + 1 = 4 = Var(V) \Longrightarrow \sigma = \sqrt{4} = 2$$

Using the result obtained in a), and the fact that the sum of normal random variables is normal we get that:

$$P(U > kV) = P(U - kV > 0) = 1 - P(U - kV \leq 0) = 1 - \Phi(0) = \boxed{0.5}$$

c) Find $P(U^2 + V^2 < 1)$.

Solution: Note: $U^2 + V^2 = (\sqrt{3}X + Y)^2 + (X - \sqrt{3}Y)^2 = 3X^2 + 2\sqrt{3}XY + Y^2 + X^2 - 2\sqrt{3}XY + 3Y = 4X^2 + 4Y^2$. The variable $R^2 = X^2 + Y^2$ where both X and Y are independent standard normal R.V.s is distributed as an exponential with parameter $\lambda = \frac{1}{2}$. Hence:

$$P(U^2 + V^2 < 1) = P(4X^2 + 4Y^2 < 1) = P\left(X^2 + Y^2 < \frac{1}{4}\right) = 1 - e^{-\frac{1}{8}} = \boxed{0.117503}$$

d) Find the conditional distribution of X given $V = v$.

Solution: By definition of V:

$$X = V + \sqrt{3}Y. \quad \text{If we are given a value } v \text{ of } V \text{ then } X = v + \sqrt{3}Y$$

This is just a linear transformation of a normal variable and hence, it is normal. Its parameters are:

$$\mu_X = E(X) = E(v + \sqrt{3}Y) = E(v) + \sqrt{3}E(Y) = v + \sqrt{3} \cdot 0 = v$$

$$\sigma_X^2 = Var(X) = Var(v + \sqrt{3}Y) = Var(\sqrt{3}Y) = (\sqrt{3})^2 Var(Y) = 3 \cdot 1 = 3 \Longrightarrow \sigma_X = \sqrt{3}$$

In short, $\boxed{X \sim Normal(\mu_X = v, \sigma_X = \sqrt{3})}$