M463 Homework 8

Enrique Areyan
June 27, 2013

(2.4) #10 Let N be a fixed large integer. Consider n independent trials, each of which is a success with prob. $1/N$. Show that if $n \approx \frac{5}{3}N$, then the chance of at least two successes is about $1/2$.

Solution: Since N is large we know that the probability of success $1/N$ is small, and that the number of trials n is also large ($n \approx \frac{5}{3}N$, so $n > N$). Therefore, we may use the Poisson Distribution to approximate the Binomial Distribution.

Let X be distributed as a binomial with probability of success $p = 1/N$ and $n =$ number of trials. Then, we want to know:

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1) = 1 - P(X = 0 \text{ OR } X = 1) = 1 - P(X = 0) - P(X = 1)$$

Using the Possion approximation with $\mu = np = n/N$:

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) \approx 1 - \frac{e^{-\frac{n}{N}} \left(\frac{n}{N}\right)^0}{0!} - \frac{e^{-\frac{n}{N}} \left(\frac{n}{N}\right)^1}{1!} = 1 - e^{-\frac{n}{N}} - \frac{n}{N} e^{-\frac{n}{N}}$$

By assumption $n \approx \frac{5}{3}N$, so $\frac{n}{N} \approx \frac{5}{3}$. Replacing in the above equation:

$$P(X \geq 2) \approx 1 - e^{-\frac{n}{N}} - \frac{n}{N} e^{-\frac{n}{N}} \approx 1 - e^{-\frac{5}{3}} - \frac{5}{3} e^{-\frac{5}{3}} = 0.496331726 \approx \boxed{\frac{1}{2}}$$

Which shows the result we wanted.