M464 - Introduction To Probability II - Homework 12

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Chapter 6

Problems

3.1 Let $\xi_n, n = 0, 1, \ldots,$ be a two state Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 \end{matrix} \\ \begin{Vmatrix} 0 & 1 \\ 1-\alpha & \alpha \end{Vmatrix} \end{array}$$

Let $\{N(t); t \geq 0\}$ be a Poisson process with parameter λ. Show that

$$X(t) = \xi_{N(t)}, \qquad t \geq 0,$$

is a two state birth and death process and determine the parameters λ_0 and μ_1 in terms of α and λ.

Solution: That $X(t)$ is a Markov chain follows from the fact that $N(.)$ is a Pois. process independent of ξ_n. Now:

$$\begin{aligned}
Pr\{X(t+h) = 1|X(t) = 0\} &= Pr\{X(h) = 1|X(0) = 0\} \\
&= Pr\{\xi_{N(h)} = 1|\xi_{N(0)} = 0\} && \text{by def. of } X(t) \\
&= P_{01}^{N(h)} && \text{by def. of transition prob.} \\
&= \sum_{i=1}^{\infty} P_{01}^{i} Pr\{N(h) = i\} && \text{law of total prob.} \\
&= \sum_{i=1}^{\infty} P_{01}^{i} e^{-\lambda h} \frac{(\lambda h)^i}{i!} && \text{Poisson process} \\
&= \sum_{i=1}^{\infty} P_{01}^{i} \left[\sum_{k=0}^{\infty} \frac{(-\lambda h)^k}{k!}\right] \frac{(\lambda h)^i}{i!} && \text{Taylor expansion of } e \\
&= \sum_{i=1}^{\infty} P_{01}^{i} \left(1 + (-\lambda h) + \left[\sum_{k=2}^{\infty} \frac{(-\lambda h)^k}{k!}\right]\right) \frac{(\lambda h)^i}{i!} && \text{Taking first two terms out of sum} \\
&= \sum_{i=1}^{\infty} P_{01}^{i} \left(1 - \lambda h + o(h)\right) \frac{(\lambda h)^i}{i!} && \text{Since } \sum_{k=2}^{\infty} \frac{(-\lambda h)^k}{k!} \text{ is } o(h) \\
&= (1 - \lambda h + o(h)) \sum_{i=1}^{\infty} P_{01}^{i} \frac{(\lambda h)^i}{i!} && \text{Since } (1 - \lambda h + o(h)) \text{ is constant w.r.t } i \\
&= (1 - \lambda h + o(h)) \left(P_{01}(\lambda h) + \sum_{i=2}^{\infty} P_{01}^{i} \frac{(\lambda h)^i}{i!}\right) && \text{Taking first term out of sum} \\
&= (1 - \lambda h + o(h)) \left(P_{01}(\lambda h) + o(h)\right) && \text{Since } \sum_{i=2}^{\infty} P_{01}^{i} \frac{(\lambda h)^i}{i!} \text{ is } o(h) \\
&= (1 - \lambda h + o(h)) \left(1(\lambda h) + o(h)\right) && \text{Since } P_{01} = 1 \\
&= \lambda h + o(h) - (\lambda h)^2 - o(h)\lambda h + o(h)\lambda h + o(h)o(h) && \text{Distributing} \\
&= \lambda h + o(h) && \text{Grouping all } o(h)
\end{aligned}$$

Hence, $Pr\{X(t+h) = 1|X(t) = 0\} = \lambda h + o(h)$ which means that $\boxed{\lambda_0 = \lambda}$, by dividing by h and letting $h \to 0$.

$$
\begin{aligned}
Pr\{X(t+h)=1|X(t)=0\} &= Pr\{X(h)=0|X(0)=1\} && \\
&= Pr\{\xi_{N(h)}=0|\xi_{N(0)}=1\} && \text{by def. of } X(t) \\
&= P_{10}^{N(h)} && \text{by def. of transition prob.} \\
&= \sum_{i=1}^{\infty} P_{10}^{i} Pr\{N(h)=i\} && \text{law of total prob.} \\
&= \sum_{i=1}^{\infty} P_{10}^{i} e^{-\lambda h}\frac{(\lambda h)^i}{i!} && \text{Poisson process} \\
&= \sum_{i=1}^{\infty} P_{10}^{i} \left[\sum_{k=0}^{\infty}\frac{(-\lambda h)^k}{k!}\right]\frac{(\lambda h)^i}{i!} && \text{Taylor expansion of } e \\
&= \sum_{i=1}^{\infty} P_{10}^{i} \left(1+(-\lambda h)+\left[\sum_{k=2}^{\infty}\frac{(-\lambda h)^k}{k!}\right]\right)\frac{(\lambda h)^i}{i!} && \text{Taking first two terms out of sum} \\
&= \sum_{i=1}^{\infty} P_{10}^{i} \left(1-\lambda h+o(h)\right)\frac{(\lambda h)^i}{i!} && \text{Since } \sum_{k=2}^{\infty}\frac{(-\lambda h)^k}{k!} \text{ is } o(h) \\
&= (1-\lambda h+o(h))\sum_{i=1}^{\infty} P_{10}^{i}\frac{(\lambda h)^i}{i!} && \text{Since } (1-\lambda h+o(h)) \text{ is constant w.r.t } i \\
&= (1-\lambda h+o(h))\left(P_{10}(\lambda h)+\sum_{i=2}^{\infty} P_{10}^{i}\frac{(\lambda h)^i}{i!}\right) && \text{Taking first term out of sum} \\
&= (1-\lambda h+o(h))\left(P_{10}(\lambda h)+o(h)\right) && \text{Since } \sum_{i=2}^{\infty} P_{10}^{i}\frac{(\lambda h)^i}{i!} \text{ is } o(h) \\
&= (1-\lambda h+o(h))\left((1-\alpha)(\lambda h)+o(h)\right) && \text{Since } P_{10}=1-\alpha \\
&= (1-\alpha)\lambda h+o(h) && \text{Grouping all } o(h)
\end{aligned}
$$

Hence, $Pr\{X(t+h)=0|X(t)=1\}=(1-\alpha)\lambda h+o(h)$ which means that $\boxed{\mu_1=(1-\alpha)\lambda}$.

3.4 *A Stop-and-Go Traveler* The velocity $V(t)$ of a stop-and-go traveler is described by the two state Markov chain whose transition probabilities are given by (3.12a-d). The distance traveled in time t is the integral of the velocity:

$$S(t)=\int_0^t V(u)du$$

Assuming that the velocity at time $t=0$ is $V(0)=0$, determine the mean of $S(t)$. Take for granted the interchange of integral and expectation in

$$E[S(t)]=\int_0^t E[V(u)]du.$$

Solution: First, let us compute the expectation of $V(t)$. Note that the random variable $V(t)=0$ or 1, since the velocity is described by the aforementioned two-state Markov chain with states 0 and 1. Moreover, we know that the $V(0)=0$, so the chain starts in state 0. Hence:

$$E[V(t)]=0\cdot Pr\{V(t)=0\}+1\cdot Pr\{V(t)=1\}=Pr\{V(t)=1\}=P_{01}(t)=\pi-\pi e^{-\tau t}$$

where $\pi=\alpha/(\alpha+\beta)$ and $\tau=\alpha+\beta$. Now we can compute the mean of $S(t)$:

$$E[S(t)]=\int_0^t E[V(u)]du=\int_0^t \pi-\pi e^{-\tau u}du=\left[\pi u+\frac{\pi}{\tau}e^{-\tau u}\right]_0^t=(\pi t+\frac{\pi}{\tau}e^{-\tau t})-(\pi 0+\frac{\pi}{\tau}e^{-\tau 0})=\boxed{\pi t+\frac{\pi}{\tau}(e^{-\tau t}-1)}$$

This functions makes intuitive sense, as t goes to infinity the mean distance traveled will also go to infinity, i.e., the total distance traveled will increase with no bound. Also, as t goes to zero the mean distance traveled will also go to zero, i.e., no distance traveled at the beginning.