M464 - Introduction To Probability II - Homework 13

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Chapter 6

Problems

4.2 Determine the stationary distribution, when it exists, for a birth and death process having constant parameters $\lambda_n = \lambda$ for $n = 0, 1, \ldots$ and $\mu_n = \mu$ for $n = 1, 2, \ldots$

Solution: By equations 4.6 and 4.7 in the book, we have:

$$\pi_j = \frac{\theta_j}{\sum\limits_{k=0}^{\infty} \theta_k} \qquad \text{where, } \theta_0 = 1 \text{ and } \theta_j = \frac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\ldots\mu_j}$$

In this case, $\theta_j = \dfrac{\lambda_0\lambda_1\cdots\lambda_{j-1}}{\mu_1\mu_2\ldots\mu_j} = \dfrac{\lambda^j}{\mu^j} = \left(\dfrac{\lambda}{\mu}\right)^j$, since the birth and death parameters are constants. Hence,

$$\sum_{k=0}^{\infty} \theta_k = \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j$$

For a limiting distribution to exists, a necessary and sufficient condition is $\sum\limits_{k=0}^{\infty} \left(\dfrac{\lambda}{\mu}\right)^j < \infty$.

Hence, it is necessary that $\left|\dfrac{\lambda}{\mu}\right| = \dfrac{\lambda}{\mu} < 1$, (rates are not negative!), from which it follows that $\lambda < \mu$. In this case:

$$\sum_{k=0}^{\infty} \theta_k = \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = \frac{1}{1 - \frac{\lambda}{\mu}} = \frac{\mu}{\mu - \lambda} \qquad \text{(sum of a geometric series of ratio } \lambda/\mu\text{)}$$

Solving for π_j:

$$\pi_j = \frac{\left(\frac{\lambda}{\mu}\right)^j}{\frac{\mu}{\mu - \lambda}} = \boxed{\left(\frac{\lambda}{\mu}\right)^j \left(\frac{\mu - \lambda}{\mu}\right)}$$

Note that the condition $\lambda < \mu$ makes sense: in average we should have more deaths than births or otherwise the population size will explote to infinity. Also note that the derived π_j's form a proper distribution because each π_j is positive between 0 and 1 and their sum adds to one as the following calculation shows:

$$\sum_{j=0}^{\infty} \pi_j = \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j \left(\frac{\mu - \lambda}{\mu}\right) = \left(\frac{\mu - \lambda}{\mu}\right) \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = \left(\frac{\mu - \lambda}{\mu}\right)\left(\frac{\mu}{\mu - \lambda}\right) = 1$$

4.3 A factory has five machines and a single repairman. The operating time until failure of a machine is an exponentially distributed random variable with parameter (rate) 0.20 per hour. The repair time of a failed machine is an exponentially distributed random variable with parameter (rate) 0.50 per hour. Up to five machines may be operating at any given time, their failures being independent of one another, but at most one machine may be in repair at any time. In the long run, what fraction of time is the repairman idle?

Solution: Let $X(t)$ = number of machines working at time t. By analogy with the *repairman model* in page 369, we have that $\langle X(t); t\rangle$ is a birth and death process where a "death" means a computer failure and a birth means that a computer has been repaired. Now, the parameters for this particular process are, for $n = 0, 1, \ldots, 5$:

$$\lambda_n = \begin{cases} 0.5 & \text{if } 0 \leq n < 5 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \mu_n = 0.2 \times min\{n, 5\} = 0.2n$$

That is, only one machine can be repaired at one time and thus, the rate of birth is 0.5 unless there are no machines to repair, i.e., $n = 5$, in which case the birth rate is zero. The rate at which machines fails is just $0.2n$, i.e., the individual rate of failure times the number of working machines.

We are interested in finding π_5, the long-run probability of all machines working which corresponds to the state where the repairman is idle. This long-run probability is the same as the fraction of time said repairman is idle. By equations 4.6 and 4.7 in the book, we have:

$$\pi_5 = \frac{\theta_5}{\sum\limits_{k=0}^{5} \theta_k}$$

Where:

k	θ_k
0	1
1	$\dfrac{\lambda_0}{\mu_1} = \dfrac{0.5}{0.2} = \dfrac{5}{2}$
2	$\dfrac{\lambda_0\lambda_1}{\mu_1\mu_2} = \dfrac{0.5^2}{0.2^2 \times 2} = \dfrac{25}{8}$
3	$\dfrac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = \dfrac{0.5^3}{0.2^3 \times 2 \times 3} = \dfrac{125}{48}$
4	$\dfrac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4} = \dfrac{0.5^4}{0.2^4 \times 2 \times 3 \times 4} = \dfrac{625}{384}$
5	$\dfrac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4}{\mu_1\mu_2\mu_3\mu_4\mu_5} = \dfrac{0.5^5}{0.2^5 \times 2 \times 3 \times 4 \times 5} = \dfrac{3125}{3840}$

Hence,

$$\sum_{k=0}^{5} \theta_k = 1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{48} + \frac{625}{384} + \frac{3125}{3840} = \frac{3840 + 9600 + 12000 + 10000 + 6250 + 3125}{3840} = \frac{44815}{3840}$$

Finally,

$$\pi_5 = \frac{\theta_5}{\sum\limits_{k=0}^{5} \theta_k} = \frac{\frac{3125}{3840}}{\frac{44815}{3840}} = \boxed{\frac{3125}{44815}} \approx 6.97\%$$

The repairman is idle about 6.97% of the time.

4.4 This problem considers a continuous time Markov chain model for the changing pattern of relationship among members in a group. The group has four members: a, b, c, and d. Each pair of the group may or may not have a certain relationship with each other. If they have the relationship, we say that they are *linked.* For example, being linked may mean that the two members are communicating with each other.
Suppose that any pair of unlinked individuals will become linked in a small time interval on length h with probability $\alpha h + o(h)$. Any pair of linked individuals will lose their link in a small time interval of length h with probability $\beta h + o(h)$. Let $X(t)$ denote the number of linked pairs of individuals in the group at time t. Then $X(t)$ is a birth and death process.

(a) Specify the birth and death parameters λ_k and μ_k for $k = 0, 1, \ldots$

Solution: There are a total of $\binom{4}{2} = 6$ possible links. Hence, for $k = 0, 1, \ldots 6$

$$\lambda_k = (6-k)\alpha \qquad \mu_k = \beta k \qquad (\text{Note: } \lambda_k = \mu_k = 0 \text{ if k} > 6)$$

That is, if there are no links, $k = 0$, then the rate of "birth" of new links is the product of the individual link rates. If all links are present, then there should be no new links. Similarly, if there are no links then there can be no "deaths" of links. Is all links are present, then the maximum rate of "unlinking" occurs.

(b) Determine the stationary distribution for the process

Solution: Again, by equations 4.6 and 4.7 in the book, we have:

$$\theta_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} = \frac{(6 \cdot 5 \cdots (6-k+1))\alpha^k}{(1 \cdot 2 \cdots k)\beta^k} = \binom{6}{k}\left(\frac{\alpha}{\beta}\right)^k \Longrightarrow \boxed{\pi_j = \frac{\binom{6}{j}\left(\frac{\alpha}{\beta}\right)^j}{\sum\limits_{k=0}^{6}\binom{6}{k}\left(\frac{\alpha}{\beta}\right)^k}} \quad \text{for } j = 0, 1, \ldots, 6$$