M464 - Introduction To Probability II - Homework 2

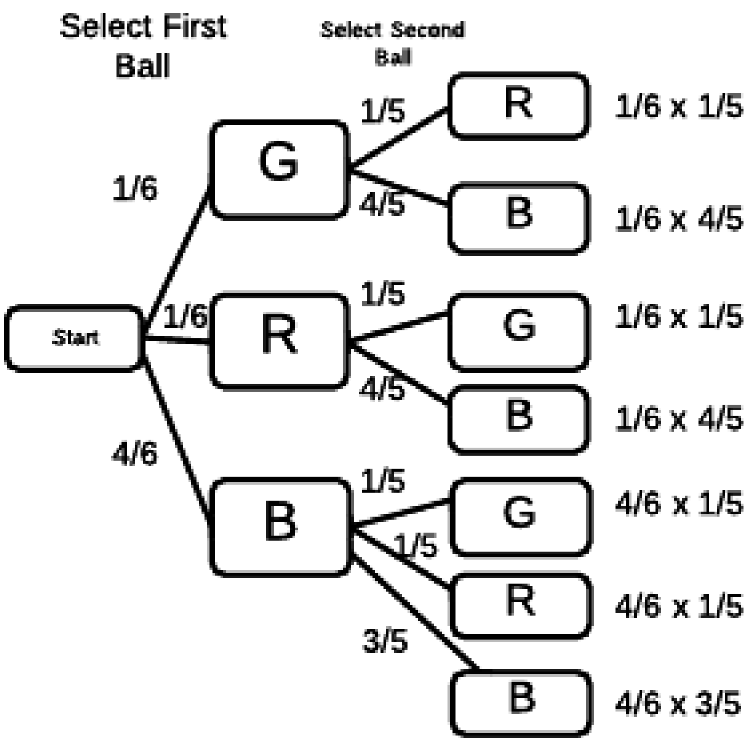
Enrique Areyan
January 30, 2014

Chapter 3

(3.1) An urn contains six tags, of which three are red and three are green. Two tags are selected from the urn. If one tag is red and the other is green, then the selected tags are discarded and two blue tags are returned to the urn. Otherwise, the selected tags are returned to the urn. This process repeats until the urn contains only blue tags. Let X_n denote the number of red tags in the urn after the nth draw, with $X_0 = 3$. Given the transition probability matrix.

Solution: First note that if $X_n = 0$ (we have no red balls), then $X_{n+1} = 0$, i.e., we will have no red balls. (State 0 is absorbing). Now, let us analyze the transition by cases. This is easily done by drawing the tree diagram for the experiment of drawing two balls depending on the number of red balls. Note that in the following diagrmasG stands for selecting a green ball and R a red ball:

$X_n = 1$. If there is only <u>one</u> red ball then the bag has one red ball, one green ball and four blue balls:

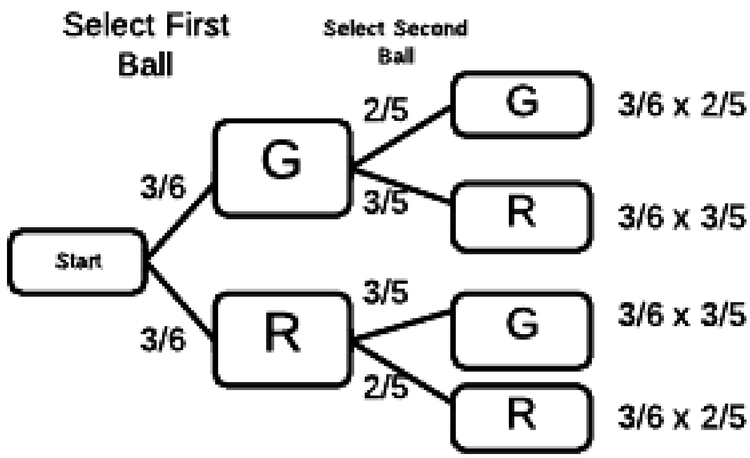


Therefore, the probability of selection one green ball and one red ball is $\frac{1}{6} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{1}{5} = 2\frac{1}{6}\frac{1}{5} = \frac{1}{15}$ This is precisely the probability of moving from state 1 to state 0 since we replace the only remaining red ball with a blue ball. The complement $1 - \frac{1}{15} = \frac{14}{15}$ is the probability of selecting something other than a green and a red ball and thus, staying in state 1.

$X_n = 2$. If there are <u>two</u> red balls then the bag has two red balls, two green balls and two blue balls: Therefore, the probability of selecting one green ball and one red ball is $\frac{2}{6} \cdot \frac{2}{5} + \frac{2}{6} \cdot \frac{2}{5} = 2\frac{2}{6} \cdot \frac{2}{5} = \frac{4}{15}$. This is precisely the probability of moving from state 2 to state 1 since we replace a red ball with a blue ball. The complement $1 - \frac{4}{15} = \frac{11}{5}$ is the probability of selecting something other than a green and a red ball and thus, staying in state 2.

I omit the tree diagram for this case to save space

$X_n = 3$. If there are <u>three</u> red balls then there are no blue balls and the bag has three red and three green balls:



Therefore, the probability of selecting one green ball and one red ball is $\frac{3}{6}\cdot\frac{3}{5}+\frac{3}{6}\cdot\frac{3}{5} = 2\frac{3}{6}\cdot\frac{3}{5} = \frac{3}{5}$. This is precisely the probability of moving from state 3 to state 2 since we replace a red ball with a blue ball. The complement $1-\frac{3}{5}=\frac{2}{5}$ is the probability of selecting something other than a green and a red ball and thus, staying in state 3.

Hence, the transition probability matrix is:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1/15 & 14/15 & 0 & 0 \\ 0 & 4/15 & 11/15 & 0 \\ 0 & 0 & 3/5 & 2/5 \end{array} \right\| \end{array}$$

(3.5) You are going to successively flip a quarter until the pattern HHT appears; that is, until you observe two successive heads followed by a tails. In order to calculate some properties of this game, you set up a Markov chain with the following states: $0, H, HH$, and HHT, where 0 represents the starting point, H represents a single observed head on the last flip, HH represents two successive heads on the last two flips, and HHT is the sequence you are looking for. Observe that if you have just tossed a tails, followed by a heads, a next toss of a tails effectively starts you over again in your quest for the HHT sequence. Set up the transition probability matrix.

Solution: First note that if you are in state HHT the game is over and so you stay in that state. In other words, the state HHT is absorbing. If you are in the starting state, then there is 1/2 chance of seeing a heads and moving to state H and 1/2 chance of seeing a tails and start over, i.e., go to state 0. In Markov chain language, these probabilities are: $P\{X_{n+1} = H|X_n = 0\} = 1/2$, and $P\{X_{n+1} = 0|X_n = 0\} = 1/2$ respectively. That takes care of the first and last rows. For the middle rows: if you are in state H, then there is 1/2 chance of seeing another head and thus, move to state HH. However, if you are in state H and immediately see a tails, then you have to start over and so you go to state 0 with probability 1/2. Finally, if you are in state HH then there is 1/2 probability of seeing a heads and staying in state HH and 1/2 probability of seeing a tail and finishing the game, i.e., moving to state HHT.
Hence, the transition probability matrix is:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ H \\ HH \\ HHT \end{array} \begin{array}{c} \begin{array}{cccc} 0 & H & HH & HHT \end{array} \\ \left\| \begin{array}{cccc} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(3.6) Two teams, A and B, are to play a best of seven series of games. Suppose that the outcomes of successive games are independent, and each is won by A with probability p and won by B with probability $1-p$. Let the state of the system be represented by the pair (a, b), where a is the number of games won by A, and b is the number of games won by B. Specify the transition probability matrix. Note that $a + b \leq 7$ and that the series ends whenever $a = 4$ or $b = 4$.

Solution: Let (i, j) and (k, l) be states in the system. If $i, j, k, l \in \{0, 1, 2, 3\}$ then a key observation is that transitions can only occur between states (i, j) and (k, l) if the difference between $k+l$ and $i+j$ is one, i.e., $(k+l)-(i+j) = 1$.

This is because states represent successive games, one played after the other. So, for example, we can't transition from state $(1,2)$ to state $(3,4)$, because there are games to be play in between. In this example we can only transition from $(1,2)$ to state $(2,2)$ in case team A wins the next game and ties the series, or $(1,3)$ in case team B wins.
If either $i = 4$ or $j = 4$, then there are no transitions. In other words, states $(4,i)$ and $(i,4)$ are absorbing for $i \in \{0,1,2,3\}$, representing the fact that the series is over and one of the teams have won.
Let us write the transition matrix in two cases:

If $i,j,k,l \in \{0,1,2,3\}$ then, either one of the teams can win the next game with probabilities given by:

$$P_{(i,j),(k,l)} = \begin{cases} p & \text{if } k = i+1 \text{ and } j = l \quad \text{(team A wins)} \\ 1-p & \text{if } l = j+1 \text{ and } i = k \quad \text{(team B wins)} \\ 0 & \text{otherwise} \end{cases}$$

Otherwise, if $i = 4$ or $j = 4$, the series is over and these are absorbing state, i.e.,:

$$P_{(i,j),(k,l)} = \begin{cases} 1 & \text{if } (i,j) = (k,l) \\ 0 & \text{otherwise} \end{cases}$$

(4.5) *(Exercise).* A coin is tossed repeatedly until either two successive heads appear or two successive tails appear. Suppose the first coin toss results in a head. Find the probability that the game ends with two successive tails.

Solution: We could model this game as a 4 or a 6 state Markov Chain. In the first case we would have states $\{HT, TH, HH, TT\}$ and in the second case states $\{H, T, HT, TH, HH, TT\}$. In the former case we would omit states H and T and think of the game as consisting of only the last two tosses of the coin. In the latter, we would include H and T which could seem as "artificial" since the game is concern only with the last two states of the coin. However, including states H and T has the advantage of including all possibilities and allowing for direct computeation and thus, I will model the game with 6 states as given by the following transition probability matrix:

$$\mathbf{P} = \begin{array}{c} \\ H \\ T \\ HT \\ TH \\ HH \\ TT \end{array} \begin{array}{c} \begin{array}{cccccc} H & T & HT & TH & HH & TT \end{array} \\ \left\| \begin{array}{cccccc} 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

States HH and TT are absorbing since these are the final states of the game. Other states follow simple rules of probability assuming the coin is fair and tosses are independent. For example, transitioning from state HT to state TT means the toss X_n results in a tail T with probability 1/2.

Having setup the transition matrix, let us perform first step analysis. First, let us use define:

$$T = min\{n \geq 0; X_n = HH \text{ or } X_n = TT\} \quad \text{- absorption time -}$$

And the following probabilities:

$$u_i = Pr\{X_T = TT | X_0 = i\}, \text{ where } i \in \{H, T, HT, TH, HH, TT\}$$

From this and the Markov property, it follows that the probabilities for the first step X_1 are:

$$\begin{aligned} Pr\{X_T = TT | X_1 = H\} &= u_H \\ Pr\{X_T = TT | X_1 = T\} &= u_T \\ Pr\{X_T = TT | X_1 = HT\} &= u_{HT} \\ Pr\{X_T = TT | X_1 = TH\} &= u_{TH} \\ Pr\{X_T = TT | X_1 = HH\} &= u_{HH} = 0 \quad \text{since state } HH \text{ is absorbing} \\ Pr\{X_T = TT | X_1 = TT\} &= u_{TT} = 1 \quad \text{since we are already in state } TT \end{aligned}$$

We wish to solve for u_H, i.e., the probability that the game ends with two successive tails TT given that the first coin toss results in a head. By first step analysis (law of total probability and conditioning on the first step), we have:

$$\begin{array}{rcl}
u_H & = & Pr\{X_T = TT|X_0 = H\} \\
& = & \sum_i Pr\{X_T = TT|X_0 = H, X_1 = i\}Pr\{X_1 = i|X_0 = H\}, \text{ where } i \in \{H, T, HT, TH, HH, TT\} \text{ law of total prob.} \\
& = & \sum_i Pr\{X_T = TT|X_1 = i\}Pr\{X_1 = i|X_0 = H\}, \text{ Markov Property} \\
& = & Pr\{X_T = TT|X_1 = H\}Pr\{X_1 = H|X_0 = H\} + Pr\{X_T = TT|X_1 = T\}Pr\{X_1 = T|X_0 = H\}+ \\
& & Pr\{X_T = TT|X_1 = HT\}Pr\{X_1 = HT|X_0 = H\} + Pr\{X_T = TT|X_1 = TH\}Pr\{X_1 = TH|X_0 = H\}+ \\
& & Pr\{X_T = TT|X_1 = HH\}Pr\{X_1 = HH|X_0 = H\} + Pr\{X_T = TT|X_1 = TT\}Pr\{X_1 = TT|X_0 = H\} \\
& = & 0u_H + 0u_T + \frac{1}{2}u_{HT} + 0u_{TH} + \frac{1}{2}u_{HH} + 0u_{TT} \quad \text{using the information on the transition matrix and previous def.} \\
& = & \frac{1}{2}u_{HT} + \frac{1}{2}u_{HH} \quad \text{simplifying} \\
& = & \frac{1}{2}u_{HT} \quad \text{since } u_{HH} = 0
\end{array}$$

Therefore, $u_H = \frac{1}{2}u_{HT}$. Proceeding in this manner, we can set up the following equations (I won't type all details to save space):

$$\begin{array}{rcl}
u_{HT} & = & \frac{1}{2}u_{TH} + \frac{1}{2} \\
u_{TH} & = & \frac{1}{2}u_{HT}
\end{array}$$

Now we can solve this simultaneous system:

$$u_{HT} = \frac{1}{2}(\frac{1}{2}u_{HT}) + \frac{1}{2} = \frac{1}{4}u_{HT} + \frac{1}{2} \Rightarrow \frac{3}{4}u_{HT} = \frac{1}{2} \Rightarrow \boxed{u_{HT} = \frac{2}{3}}$$

Lastly, $u_H = \frac{1}{2}u_{HT} = \frac{1}{2}\frac{2}{3} = \boxed{\frac{1}{3}}$. Hence, there is a $\frac{1}{3}$ probability that the game ends with two successive tails TT given that the first coin toss results in a head

(4.2) A zero-seeking device operates as follows: If it is in state m at time n, then at time $n+1$, its position is uniformly distributed over the states $0, 1, \ldots, m-1$. Find the expected time until the device first hits zero starting from state m.

Solution: The device can be modeled as a Markov Chain with the following transition probability matrix is:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ m-1 \\ m \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & m-1 & m \end{array} \\
\left\| \begin{array}{ccccccc}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & \cdots & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1/m-1 & 1/m-1 & 1/m-1 & 1/m-1 & \cdots & 0 & 0 \\
1/m & 1/m & 1/m & 1/m & \cdots & 1/m & 0
\end{array} \right\|
\end{array}$$

Let $T = min\{n \geq 0 : X_n = 0\}$. We wish to compute $E[T|X_0 = m]$. Let $v_i = E[T|X_0 = i]$, for $i = 0, 1, 2, \ldots, m-1, m$. Note that $v_0 = 0, v_1 = 1$, i.e., if we are in state 0 there is no wait time, and if we are in state 1 we are certain to be in state 0 the next time period. By first step analysis, we have that:

$$\begin{array}{rcl}
v_2 & = & 1 + v_0\frac{1}{2} + v_1\frac{1}{2} \\
v_3 & = & 1 + v_0\frac{1}{3} + v_1\frac{1}{3} + v_2\frac{1}{3} \\
v_4 & = & 1 + v_0\frac{1}{4} + v_1\frac{1}{4} + v_2\frac{1}{4} + v_3\frac{1}{4} \\
& \vdots & \\
v_m & = & 1 + v_0\frac{1}{m} + v_1\frac{1}{m} + v_2\frac{1}{m} + \cdots + v_{m-1}\frac{1}{m}
\end{array}$$

These are $m+1$ equations (including v_0 and v_1), which can be solved simultaneously:

$$\begin{aligned}
v_2 &= 1 + v_0\tfrac{1}{2} + v_1\tfrac{1}{2} = 1 + 0\tfrac{1}{2} + 1\tfrac{1}{2} = 1 + \tfrac{1}{2} \\
v_3 &= 1 + v_0\tfrac{1}{3} + v_1\tfrac{1}{3} + v_2\tfrac{1}{3} = 1 + 0\tfrac{1}{3} + 1\tfrac{1}{3} + \tfrac{1}{3}(1+\tfrac{1}{2}) = 1 + \tfrac{1}{3} + (\tfrac{1}{3}+\tfrac{1}{6}) = 1 + \tfrac{1}{2} + \tfrac{1}{3} \\
v_4 &= 1 + v_0\tfrac{1}{4} + v_1\tfrac{1}{4} + v_2\tfrac{1}{4} + v_3\tfrac{1}{4} = 1 + 0\tfrac{1}{4} + 1\tfrac{1}{4} + \tfrac{1}{4}(1+\tfrac{1}{2}) + \tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3}) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} \\
&\vdots \\
v_m &= 1 + v_0\tfrac{1}{m} + v_1\tfrac{1}{m} + v_2\tfrac{1}{m} + \cdots + v_{m-1}\tfrac{1}{m} = \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{m}
\end{aligned}$$

Therefore, $\boxed{v_m = E[T|X_0 = m] = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{m} = \sum_{k=1}^{m} \frac{1}{k}}$. Note that this result can be proved by induction.

(4.12) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

and is known to start in state $X_0 = 0$. Eventually, the process will end up in state 2. What is the probability that when the process moves into state 2, it does so from state 1?

Solution: Following the hint, let $T = min\{n \geq 0; X_n = 2\}$, and let

$$z_i = Pr\{X_{T-1} = 1|X_0 = i\} \qquad \text{for } i = 0, 1.$$

By First Step Analysis:

$$\begin{aligned}
z_0 &= Pr\{X_{T-1} = 1|X_0 = 0\} \\
&= \sum_i Pr\{X_{T-1} = 1|X_0 = 0, X_1 = i\}Pr\{X_1 = i|X_0 = 0\}, \text{ where } i \in \{0,1,2\} \text{ law of total prob.} \\
&= \sum_i Pr\{X_{T-1} = 1|X_0 = i\}Pr\{X_1 = i|X_0 = 0\}, \text{ Markov Property} \\
&= Pr\{X_{T-1} = 1|X_1 = 0\}Pr\{X_1 = 0|X_0 = 0\} + Pr\{X_{T-1} = 1|X_1 = 1\}Pr\{X_1 = 1|X_0 = 0\} + \\
&\quad Pr\{X_{T-1} = 1|X_1 = 2\}Pr\{X_1 = 2|X_0 = 0\} \\
&= 0.3z_0 + 0.2z_1 + 0 \cdot 0.5 \quad \text{using the information on the transition matrix and previous def. also see (*)} \\
&= 0.3z_0 + 0.2z_1 \quad \text{simplifying}
\end{aligned}$$

(*) Note that since we know that $X_0 = 0$, and we condition on the event that $X_1 = 2$, then $T = 1$ and thus:

$Pr\{X_{T-1} = 1|X_1 = 2\} = Pr\{X_{1-1} = 1|X_1 = 2\} = Pr\{X_0 = 1|X_1 = 2\} = 0$, since $X_0 = 0$ and thus it is impossible that $X_0 = 1$

Hence, $z_0 = 0.3z_0 + 0.2z_1 \Rightarrow z_0 = \frac{2}{7}z_1$. Now for z_1:

$$\begin{aligned}
z_1 &= Pr\{X_{T-1} = 1|X_0 = 1\} \\
&= \sum_i Pr\{X_{T-1} = 1|X_0 = 1, X_1 = i\}Pr\{X_1 = i|X_0 = 1\}, \text{ where } i \in \{0,1,2\} \text{ law of total prob.} \\
&= \left(\sum_{i \in \{0,1\}} Pr\{X_{T-1} = 1|X_0 = i\}Pr\{X_1 = i|X_0 = 0\} \right) + Pr\{X_{T-1} = 1|X_0 = 1, X_1 = 2\}Pr\{X_1 = 2|X_0 = 1\}, \text{ M. Property} \\
&= 0.5z_0 + 0.1z_1 + 1 \cdot 0.4 \quad \text{using the information on the transition matrix and previous def. also see (**)}
\end{aligned}$$

(**) Note that $Pr\{X_{T-1} = 1|X_0 = 1, X_1 = 2\} = 1$, since the event we are calculating its probability has already happened. To see why, note that $X_1 = 2$ imply that $T = 1$, and so:

$$Pr\{X_{T-1} = 1|X_0 = 1, X_1 = 2\} = Pr\{X_0 = 1|X_0 = 1, X_1 = 2\} = 1$$

Hence, $z_1 = 0.5z_0 + 0.1z_1 + 0.4$. Now we can solve this simultaneous system, replacing the first into the second equation:

$$z_1 = 0.5(\frac{2}{7}z_1) + 0.1z_1 + 0.4 = z_1(\frac{1}{7} + \frac{1}{10}) + 0.4 = \frac{17}{70}z_1 + 0.4 \Rightarrow (1 - \frac{17}{70})z_1 = 0.4 \Rightarrow z_1 = \frac{28}{53}$$

$z_0 = \frac{2}{7} \cdot \frac{28}{53} = \boxed{\frac{8}{53}}$ = probability that when the process moves into state 2, it does so from state 1, knowing that $X_0 = 0$

(4.15) A simplified model for the spread of a rumor goes this way: There are $N = 5$ people in a group of friends, of which some have heard the rumor and the others have not. During any single period of time, two people are selected at random from the group and assumed to interact. The selection is such that an encounter between any pair of friends is just as likely as between any other pair. If one of these persons has heard the rumor and the other has not, then with probability $\alpha = 0.1$ the rumor is transmitted. Let X_n denote the number of friends who have heard the rumor at the end of the nth period. Assuming that the process begins at time 0 with a single person knowing the rumor, what is the mean time that it takes for everyone to hear it?

Solution: First, let us model this problem as a 5 state Markov chain with the following transition probability matrix:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\| \begin{array}{ccccc} 24/15 & 1/25 & 0 & 0 & 0 \\ 0 & 47/50 & 3/50 & 0 & 0 \\ 0 & 0 & 47/50 & 3/50 & 0 \\ 0 & 0 & 0 & 24/25 & 1/25 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

These probabilities were obtained as follow: first, state 5 is an absorbing state since all the friends know the rumor. Since in our model the rumor can't be "unknown" or "forgotten", only increments of one or no increments at all are allowed. In state 1 we can spread the rumor to another friend with if we selected the friend that knows the rumor and another friend times the probability $\alpha = 0.1$ of spreading the rumor. This experiment follows a hypergeometric distribution where $k = 1$ is the number of successes (I will consider a success selecting the only friend that knows the rumor). Hence, the probability of selecting the friend that knows the rumor and another friend is

$$\frac{\binom{1}{1}\binom{4}{1}}{\binom{5}{2}} = \frac{2}{5} = \text{ probability of selecting the friend that knows the rumor plus another friend}$$

Now, selecting this couple is not enough to spread the rumor, i.e., the rumor might or might not be spread with probability $\alpha = 0.1$. Hence $P_{1,2} = \frac{1}{10}\frac{2}{5} = \frac{1}{25}$, i.e., the probability of spreading the rumor to a friend given that only one friend knows the rumor. Moreover, the complement probability is the probability of not spreading the rumor given that only one friend knows the rumor, i.e., $P_{1,1} = 1 - \frac{1}{25} = \frac{24}{25}$.
Likewise, we can complete the table. Note that by symmetry of the binomial coefficient $P_{1,2} = P_{4,5}$ and $P_{2,3} = P_{3,4}$. Therefore, $P_{1,1} = 1 - P_{1,2} = 1 - P_{4,5} = P_{4,4}$ and $P_{2,2} = 1 - P_{2,3} = 1 - P_{3,4} = P_{3,3}$. So we need only to compute:

$$\frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{3}{5} = \text{ probability of selecting the friend that knows the rumor plus another friend}$$

Considering the probability of spreading the rumor we have: $P_{2,3} = \frac{3}{5}\frac{1}{10} = \frac{3}{50}$. Then, $P_{3,3} = 1 - \frac{3}{50} = \frac{47}{50}$.

Now, let us perform first step analysis. Let $T = min\{n \geq 0; X_n = 5\}$ and $v_i = E[T|X_0 = i]$ for $i = 1, 2, 3, 4$. Then:

$$v_1 = 1 + \tfrac{24}{25}v_1 + \tfrac{1}{25}v_2 \quad \Rightarrow \quad v_1 = 25 + v_2$$

$$v_2 = 1 + \tfrac{47}{50}v_2 + \tfrac{3}{50}v_3 \quad \Rightarrow \quad v_2 = \tfrac{50}{3} + v_3$$

$$v_3 = 1 + \tfrac{47}{50}v_3 + \tfrac{3}{50}v_4 \quad \Rightarrow \quad v_3 = \tfrac{50}{3} + v_4$$

$$v_4 = 1 + \tfrac{24}{25}v_4 \quad \Rightarrow \quad v_4 = 25$$

First note that $v_5 = 0$, since the expected time to spread the rumor given that it is already spread is 0.
Also, note that in each equation a 1 guarantees that we have to wait at least one more time period to further spread the rumor. From the last equation we get that $v_4 = 25$. Replacing this values in equation 3 we get: $v_3 = \frac{50}{3} + 24 = \frac{125}{3}$, and replacing values one after the other we obtain: $v_2 = \frac{50}{3} + \frac{125}{3} = \frac{175}{3}$ and $v_1 = 25 + \frac{175}{3} = \frac{250}{3}$, and so, assuming that the process begins at time 0 with a single person knowing the rumor, the mean time that it takes for everyone to hear it is:

$$\boxed{v_1 = \frac{250}{3}}$$

.