M464 - Introduction To Probability II - Homework 3

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Chapter 3

(5.1) As a special case of the successive maxima Markov chain whose transition probabilities are given in equation (5.5), consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \left\| \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 + a_1 & a_2 & a_3 \\ 0 & 0 & a_0 + a_1 + a_2 & a_3 \\ 0 & 0 & 0 & 1 \end{array} \right\|$$

Starting in state 0, show that the mean time until absorption is $v_0 = 1/a_3$

Solution: By *first step analysis*, let $T = min\{n \geq 0; X_n = 3\}$ and $v_i = E[T|X_0 = i]$ for $i = 0, 1, 2$. Note that $v_3 = 0$, since the expected time to reach the absorbing state given that we are already there in the first time is 0. As usual, we setup the equations:

$$\begin{aligned} v_0 &= 1 + a_0 v_0 + a_1 v_1 + a_2 v_2 + a_3 v_3 \\ v_1 &= 1 + 0 \cdot v_0 + (a_0 + a_1) v_1 + a_2 v_2 + a_3 v_3 \\ v_2 &= 1 + 0 \cdot v_0 + 0 \cdot v_1 + (a_0 + a_1 + a_2) v_2 + a_3 v_3 \end{aligned}$$

As usual, we add 1 to each equation because we expected to take at least one more step before reaching the absorption state. Now, simplifying the equations and using the fact that $v_3 = 0$:

$$\begin{aligned} v_0 &= 1 + a_0 v_0 + a_1 v_1 + a_2 v_2 \\ v_1 &= 1 + (a_0 + a_1) v_1 + a_2 v_2 \\ v_2 &= 1 + (a_0 + a_1 + a_2) v_2 \Longrightarrow [1 - (a_0 + a_1 + a_2)] v2 = 1 \Longrightarrow v_2 = \frac{1}{1-(a_0+a_1+a_2)} \end{aligned}$$

Replacing v_2 into v_1:

$$v_1 = 1 + (a_0 + a_1) v_1 + \frac{a_2}{1-(a_0+a_1+a_2)} \Longrightarrow [1 - (a_0 + a_1)] v_1 = \frac{1-(a_0+a_1)}{1-(a_0+a_1+a_2)} \Longrightarrow v_1 = \frac{1}{1-(a_0+a_1+a_2)}$$

Replacing v_1 into v_0:

$$v_0 = 1 + a_0 v_0 + \frac{a_1}{1-(a_0+a_1+a_2)} + \frac{a_2}{1-(a_0+a_1+a_2)} \Longrightarrow [1 - a_0] v_0 = 1 + \frac{a_1+a_2}{1-(a_0+a_1+a_2)} = \frac{1-a_0}{1-(a_0+a_1+a_2)} \Longrightarrow v_0 = \frac{1}{1-(a_0+a_1+a_2)}$$

Since each row of the Markov chain must add up to one (probability distribution), we have that $a_0 + a_1 + a_2 + a_3 = 1 \Longrightarrow a_3 = 1 - (a_0 + a_1 + a_2)$. Replacing this value in v_0, we get:

$$\boxed{v_0 = \frac{1}{a_3}}$$

(5.2) A component of a computer has an active life, measured in discrete units, that is a random variable T, where $Pr\{T = k\} = a_k$ for $k = 1, 2, \ldots$ Suppose one starts with a fresh component, and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n. Then $\{X_n\}$ is a success runs Markov chain.

a) Specify the probabilities p_i and q_i.

b) A "planned replacement" policy calls for replacing the component upon its failure or upon its reaching age N, whichever occurs first. Specify the success runs probabilities p_i and q_i under the planned replacement policy.

Solution:

a) Let $p_i = P_{i,0} =$ the probability of the component in service failing given that it has age i. Then, we can compute p_i as a conditional probability that the age of the component is exactly $i + 1$ (and thus, it will have to be replaced in the next time period) given that it has age i. These probabilities are given by the random variable T, and so we can write:

$$p_i = P_{i,0} = Pr\{\text{component with age } i \text{ fails}\} = Pr\{\text{component with age } i \text{ has a life of } i \text{ units}\} = Pr\{T = i+1 | T \geq i+1\}$$

Now we can compute this probability:

$$
\begin{aligned}
\frac{Pr\{T = i+1, T \geq i+1\}}{Pr\{T \geq i\}} &= \frac{Pr\{T = i+1\}}{Pr\{T \geq i+1\}} && \text{since the event } T = i+1 \text{ is a subset of } T \geq i+1 \\
&= \frac{a_{i+1}}{Pr\{T = i+1 \text{ OR } T = i+2 \text{ OR } \cdots\}} && \text{by definition of } T \\
&= \frac{a_{i+1}}{Pr\{T = i+1\} + Pr\{T = i+2\} + \cdots} && \text{disjoint events} \\
&= \frac{a_{i+1}}{a_{i+1} + a_{i+2} + \cdots} && \text{by definition of } T
\end{aligned}
$$

Therefore: $\boxed{p_i = \frac{a_{i+1}}{a_{i+1} + a_{i+2} + \cdots}}$. Finally, q_i is the complement of p_i, i.e., q_i = probability of the component in service lasts one more time period given that it has age i. This is easily computed by rules of probability: $\boxed{q_i = 1 - p_i = 1 - \frac{a_{i+1}}{a_{i+1} + a_{i+2} + \cdots}}$.

b) Given the "planned replacement" policy, our Markov chain can be modeled with N states (states 0 up to $N-1$) as follow:

$$
\mathbf{P} = \begin{array}{c|cccccccc|}
 & 0 & 1 & 2 & 3 & 4 & \cdots & N-1 \\
0 & p_0 & q_0 & 0 & 0 & 0 & \cdots & 0 \\
1 & p_1 & 0 & q_1 & 0 & 0 & \cdots & 0 \\
2 & p_2 & 0 & 0 & q_2 & 0 & \cdots & 0 \\
3 & p_3 & 0 & 0 & 0 & q_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
N-2 & p_{N-2} & 0 & 0 & 0 & 0 & \cdots & q_{N-2} \\
N-1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}
$$

So, we truncate the success runs Markov chain to allow up to N states. Once in state $N-1$ we must replace the component in service, even if it is still working, and hence we go from state $N-1$ to state 0 deterministically, i.e., $P_{N-1,0} = 1$. Note also that $p_i + q_i = 1$ for $i = 0, 1, \ldots, N-2$. Assuming the same distribution for the random variable T, the values of p_i and q_i for $i = 0, 1, \ldots, N-1$ are the same as those in a).

(5.4) Martha has a fair die with the usual six sides. She throws the die and records the number. She throws the die again and adds the second number to the first. She repeats this until the cumulative sum of all the tosses first exceeds 10. What is the probability that she stops at a cumulative sum of 13?

Solution: Consider the *partial sums* Markov Chain with states $0, 1, 2, \ldots, 16$ with transition probability:

$$
\mathbf{P} = \begin{array}{c|ccccccccccccccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & & & \vdots & & & & & & & & \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & & & \vdots & & & & & & & & \\
16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

This chain models the game played by Martha. Note that states 11 through 16 are absorbing since the game ends when the cumulative sum of all tosses first exceeds 10. Moreover, for states 0 through 10, we can only move in increments of 1 through 6 corresponding to tosses of the dice, all with equal probability 1/6.

Now, let us perform *first step analysis*. As usual, let $T = min\{n \geq 0; 11 \leq X_n \leq 16\}$, i.e., absorbing time, and let

$u_i = Pr\{X_T = 13 | X_0 = i\}$ for $i = 0, 1, \ldots, 10$. Then, $u_{13} = 1$ and $u_j = 0$ for $j \in \{11, 12, 14, 15, 16\}$. From this setup we obtaining the equations:

$$u_i = \sum_{j=i+1}^{i+6} \frac{1}{6} u_j, \text{ for } i = 0, 1, \ldots, 10$$

We can solve this simultaneous system by back substituting from the last equation to the first:

$$u_{10} = \sum_{j=11}^{16} \tfrac{1}{6}u_j = \tfrac{1}{6}u_{11} + \tfrac{1}{6}u_{12} + \tfrac{1}{6}u_{13} + \tfrac{1}{6}u_{14} + \tfrac{1}{6}u_{15} + \tfrac{1}{6}u_{16} = \tfrac{1}{6}0 + \tfrac{1}{6}0 + \tfrac{1}{6}1 + \tfrac{1}{6}0 + \tfrac{1}{6}0 + \tfrac{1}{6}0 = \tfrac{1}{6}$$

$$u_9 = \sum_{j=10}^{15} \tfrac{1}{6}u_j = \tfrac{1}{6}u_{10} + \tfrac{1}{6}u_{11} + \tfrac{1}{6}u_{12} + \tfrac{1}{6}u_{13} + \tfrac{1}{6}u_{14} + \tfrac{1}{6}u_{15} = \tfrac{1}{6}\tfrac{1}{6} + \tfrac{1}{6}0 + \tfrac{1}{6}0 + \tfrac{1}{6}1 + \tfrac{1}{6}0 + \tfrac{1}{6}0 = \tfrac{1}{36} + \tfrac{1}{6} = \tfrac{7}{36}$$

$$u_8 = \sum_{j=9}^{14} \tfrac{1}{6}u_j = \tfrac{1}{6}u_9 + \tfrac{1}{6}u_{10} + \tfrac{1}{6}u_{11} + \tfrac{1}{6}u_{12} + \tfrac{1}{6}u_{13} + \tfrac{1}{6}u_{14} = \tfrac{1}{6}\tfrac{7}{36} + \tfrac{1}{6}\tfrac{1}{6} + \tfrac{1}{6}0 + \tfrac{1}{6}0 + \tfrac{1}{6}1 + \tfrac{1}{6}0 = \tfrac{7}{216} + \tfrac{1}{36} + \tfrac{1}{6} = \tfrac{7+6+36}{216} = \tfrac{49}{216}$$

$$u_7 = \sum_{j=8}^{13} \tfrac{1}{6}u_j = \tfrac{1}{6}u_8 + \tfrac{1}{6}u_9 + \tfrac{1}{6}u_{10} + \tfrac{1}{6}u_{11} + \tfrac{1}{6}u_{12} + \tfrac{1}{6}u_{13} = \tfrac{1}{6}\tfrac{49}{216} + \tfrac{1}{6}\tfrac{7}{36} + \tfrac{1}{6}\tfrac{1}{6} + \tfrac{1}{6}0 + \tfrac{1}{6}0 + \tfrac{1}{6}1 = \tfrac{49}{1296} + \tfrac{7}{216} + \tfrac{1}{36} + \tfrac{1}{6} = \tfrac{49+42+36+216}{1296} = \tfrac{343}{1296}$$

$$u_6 = \sum_{j=7}^{12} \tfrac{1}{6}u_j = \tfrac{1}{6}u_7 + \tfrac{1}{6}u_8 + \tfrac{1}{6}u_9 + \tfrac{1}{6}u_{10} + \tfrac{1}{6}u_{11} + \tfrac{1}{6}u_{12} = \tfrac{1}{6}\tfrac{343}{1296} + \tfrac{1}{6}\tfrac{49}{216} + \tfrac{1}{6}\tfrac{7}{36} + \tfrac{1}{6}\tfrac{1}{6} + \tfrac{1}{6}0 + \tfrac{1}{6}0 = \tfrac{343}{7776} + \tfrac{49}{1296} + \tfrac{7}{216} + \tfrac{1}{36} = \tfrac{343+294+252+216}{7776} = \tfrac{1105}{7776}$$

$$u_5 = \sum_{j=6}^{11} \tfrac{1}{6}u_j = \tfrac{1}{6}\tfrac{1105}{7776} + \tfrac{1}{6}\tfrac{343}{1296} + \tfrac{1}{6}\tfrac{49}{216} + \tfrac{1}{6}\tfrac{7}{36} + \tfrac{1}{6}\tfrac{1}{6} + \tfrac{1}{6}0 = \tfrac{1105}{46656} + \tfrac{343}{7776} + \tfrac{49}{1296} + \tfrac{7}{216} + \tfrac{1}{36} = \tfrac{1105+2058+1764+1512+1296}{46656} = \tfrac{7735}{46656}$$

$$u_4 = \sum_{j=5}^{10} \tfrac{1}{6}u_j = \tfrac{1}{6}\tfrac{7735}{46656} + \tfrac{1}{6}\tfrac{1105}{7776} + \tfrac{1}{6}\tfrac{343}{1296} + \tfrac{1}{6}\tfrac{49}{216} + \tfrac{1}{6}\tfrac{7}{36} + \tfrac{1}{6}\tfrac{1}{6} = \tfrac{7735}{279936} + \tfrac{1105}{46656} + \tfrac{343}{7776} + \tfrac{49}{1296} + \tfrac{7}{216} + \tfrac{1}{36} = \tfrac{7735+6630+12348+10584+9072+7776}{279936} = \tfrac{54145}{279936}$$

$$u_3 = \sum_{j=4}^{9} \tfrac{1}{6}u_j = \tfrac{1}{6}\tfrac{54145}{279936} + \tfrac{1}{6}\tfrac{7735}{46656} + \tfrac{1}{6}\tfrac{1105}{7776} + \tfrac{1}{6}\tfrac{343}{1296} + \tfrac{1}{6}\tfrac{49}{216} + \tfrac{1}{6}\tfrac{7}{36} = \tfrac{54145}{1679616} + \tfrac{7735}{279936} + \tfrac{1105}{46656} + \tfrac{343}{7776} + \tfrac{49}{1296} + \tfrac{7}{216} = \tfrac{54145+46410+39780+74088+63504+54432}{1679616} = \tfrac{332359}{1679616}$$

$$u_2 = \sum_{j=3}^{8} \tfrac{1}{6}u_j = \tfrac{1}{6}\tfrac{332359}{1679616} + \tfrac{1}{6}\tfrac{54145}{279936} + \tfrac{1}{6}\tfrac{7735}{46656} + \tfrac{1}{6}\tfrac{1105}{7776} + \tfrac{1}{6}\tfrac{343}{1296} + \tfrac{1}{6}\tfrac{49}{216} = \tfrac{332359}{10077696} + \tfrac{54145}{1679616} + \tfrac{7735}{279936} + \tfrac{1105}{46656} + \tfrac{343}{7776} + \tfrac{49}{1296}$$

$$= \tfrac{332359+324870+278460+238680+444528+381024}{10077696} = \tfrac{1999921}{10077696}$$

$$u_1 = \sum_{j=2}^{7} \tfrac{1}{6}u_j = \tfrac{1}{6}\tfrac{1999921}{10077696} + \tfrac{1}{6}\tfrac{332359}{1679616} + \tfrac{1}{6}\tfrac{54145}{279936} + \tfrac{1}{6}\tfrac{7735}{46656} + \tfrac{1}{6}\tfrac{1105}{7776} + \tfrac{1}{6}\tfrac{343}{1296} = \tfrac{1999921}{60466176} + \tfrac{332359}{10077696} + \tfrac{54145}{1679616} + \tfrac{7735}{279936} + \tfrac{1105}{46656} + \tfrac{343}{7776}$$

$$= \tfrac{1999921+1994154+1949220+1670760+1432080+2667168}{60466176} = \tfrac{11713303}{60466176}$$

$$u_0 = \sum_{j=1}^{6} \tfrac{1}{6}u_j = \tfrac{1}{6}\tfrac{11713303}{60466176} + \tfrac{1}{6}\tfrac{1999921}{10077696} + \tfrac{1}{6}\tfrac{332359}{1679616} + \tfrac{1}{6}\tfrac{54145}{279936} + \tfrac{1}{6}\tfrac{7735}{46656} + \tfrac{1}{6}\tfrac{1105}{7776} = \tfrac{11713303}{362797056} + \tfrac{1999921}{60466176} + \tfrac{332359}{10077696} + \tfrac{54145}{1679616} + \tfrac{7735}{279936} + \tfrac{1105}{46656}$$

$$= \tfrac{11713303+11999526+11964924+11695320+10024560+8592480}{362797056} = \tfrac{65990113}{362797056}$$

Hence, there is a $\boxed{u_0 = \frac{65990113}{362797056} = 0.1818926364165420}$ probability that Martha stops at a cumulative sum of 13.

(6.1) Consider the random walk Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.1 & 0 & 0.9 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

Starting in state 1, determine the mean time until absorption.

Solution: By *first step analysis*, let $T = min\{n \geq 0; X_n = 0 \text{ or } X_n = 3\}$ and $v_i = E[T | X_0 = i]$ for $i = 1, 2$. Note that, $v_0 = v_3 = 0$ (already absorbed). We can find v_1 by solving the system:

$$\begin{aligned} v_1 &= 1 + 0.3v_0 + 0v_1 + 0.7v_2 + 0v_3 \\ v_2 &= 1 + 0v_0 + 0.1v_1 + 0v_2 + 0.9v_3 \end{aligned}$$

As usual, a one guarantees we will take at least one more step towards absorption. We can simplify these equations:

$$\begin{aligned} v_1 &= 1 + 0.7v_2 \\ v_2 &= 1 + 0.1v_1 \end{aligned}$$

Replacing the second equation into the first: $v_1 = 1 + 0.7(1 + 0.1v_1) = 1 + 0.7 + 0.07v_1 \Longrightarrow (1 - 0.07)v_1 = 1.7 \Longrightarrow v_1 = \frac{1.7}{0.93}$, and thus,

$$\boxed{v_1 = \frac{170}{93}}$$

(6.2) Consider the Markov chain $\{X_n\}$ whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} \alpha & 0 & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta = 1$. Determine the mean time to reach state 3 starting from state 0. That is, find $E[T|X_0 = 0]$, where $T = min\{n \geq 0; X_n = 3\}$

Solution: By *first step analysis*, let $v_i = E[T|X_0 = i]$ for $i = 0, 1, 2$. Note that $v_3 = 0$ (already absorbed). Then,

$$\begin{aligned} v_0 &= 1 + \alpha v_0 + 0v_1 + \beta v_2 + 0v_3 \\ v_1 &= 1 + \alpha v_0 + 0v_1 + 0v_2 + \beta v_3 \\ v_2 &= 1 + \alpha v_0 + \beta v_1 + 0v_2 + 0v_3 \end{aligned}$$

As usual, a one guarantees we will take at least one more step towards absorption. We can simplify these equations:

$$\begin{aligned} v_0 &= 1 + \alpha v_0 + \beta v_2 \\ v_1 &= 1 + \alpha v_0 \\ v_2 &= 1 + \alpha v_0 + \beta v_1 \end{aligned}$$

Replacing v_1 into v_2:

$$v_2 = 1 + \alpha v_0 + \beta[1 + \alpha v_0] \Longrightarrow v_2 = 1 + \beta + [\alpha + \alpha\beta]v_0$$

Replacing v_2 into v_0:

$$v_0 = 1 + \alpha v_0 + \beta(1 + \beta + [\alpha + \alpha\beta]v_0)$$

Finally, we can solve for v_0:

$$\begin{aligned} v_0 &= 1 + \alpha v_0 + \beta(1 + \beta + [\alpha + \alpha\beta]v_0) \\ &= 1 + \alpha v_0 + \beta + \beta^2 + \alpha\beta v_0 + \alpha\beta^2 v_0) \\ &= 1 + \beta + \beta^2 + [\alpha + \alpha\beta + \alpha\beta^2]v_0 \\ &\Longrightarrow \\ v_0 &= \tfrac{1+\beta+\beta^2}{1-\alpha-\alpha\beta-\alpha\beta^2} \\ &= \tfrac{1+\beta+\beta^2}{\beta-\alpha\beta-\alpha\beta^2} && \text{since } 1 - \alpha = \beta \\ &= \tfrac{1+\beta+\beta^2}{\beta(1-\alpha-\alpha\beta)} && \text{grouping } \beta \\ &= \tfrac{1+\beta+\beta^2}{\beta(\beta-\alpha\beta)} && \text{since } 1 - \alpha = \beta \\ &= \tfrac{1+\beta+\beta^2}{\beta^2(1-\alpha)} && \text{grouping } \beta \\ &= \tfrac{1+\beta+\beta^2}{\beta^3} && \text{since } 1 - \alpha = \beta \end{aligned}$$

Hence $\boxed{v_0 = \dfrac{1 + \beta + \beta^2}{\beta^3}}$, where $0 < \beta < 1$. Note that this result makes sense: if β is very close to 1, then v_0 is very close to 3, i.e., the mean time is close to 3, which would be going from state 0 to state 1 to state 2 and finally to 3. If β is very small, then it could take an arbitrarily large mean time to reach state 3.