M464 - Introduction To Probability II - Homework 5

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February 20, 2014

Chapter 4

(2.1) Consider a discrete-time periodic review inventory model (see III, Section 3.1), and let ξ_n be the total demand in period n. Let X_n be the inventory quantity on hand at the end of period n. Instead of following an (s, S) policy, a (q, Q) policy will be used: If the stock level at the end of a period is less than or equal to $q = 2$ units, then $Q = 2$ additional units will be ordered and will be available at the beginning of the next period. Otherwise, no ordering will take place. This is a (q, Q) policy with $q = 2$ and $Q = 2$. Assume that demand that is not filled in a period is lost (no back ordering). *Note:* We will assume instead that demand will be filled as much as possible.

(a) Suppose that $X_0 = 4$, and that the period demands turn out to be $\xi_1 = 3, \xi_2 = 4, \xi_3 = 0, \xi_4 = 2$. What are the end-of-period stock levels for periods $n = 1, 2, 3, 4$?

Solution: $X_0 = 4$, which is bigger than 2, so no additional units are ordered. $\xi_1 = 3$ and there is enough product to fulfill this demand, so $X_1 = X_0 - \xi_1 = 4 - 3 = 1$. The stock falls below 2 so we order 2 additional units that are available at the beginning of the next period. So the product at hand at the beginning of period 2 is $1+2 = 3$, which is not enough to fulfill the entire demand $\xi_2 = 4$, so we fulfill as much as possible i.e., 3 units and thus, $X_2 = 0$ the availability of product at the end of period 2. This quantity is less than 2, so two additional units are ordered. The demand in this period is $\xi_3 = 0$, thus $X_3 = 2$. Finally, we need to order new units since our current stock is equal to 2, and we can fulfill the demand for the next period $\xi_4 = 2$ thus having $X_4 = 4-\xi_2 = 4-2 = 2$ units at the end of period 4.

In summary, the end-of period stock levels for $n = 1, 2, 3, 4$, are $X_1 = 1, X_2 = 0, X_3 = 2, X_4 = 2$.

(b) Suppose that $\xi_1, \xi_2, \ldots$ are independent random variables, each having probability distribution where

$k =$	0	1	2	3	4
$Pr\{\xi = k\} =$	0.1	0.3	0.3	0.2	0.1

Then $X_0, X_1, \ldots$ is a Markov chain. Determine the transition probability distribution, and the limiting distribution.

Solution: The *transition probability distribution* is:

$$\mathbf{P} = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ 0 & 0.6 & 0.3 & 0.1 & 0 & 0 \\ 1 & 0.3 & 0.3 & 0.3 & 0.1 & 0 \\ 2 & 0.1 & 0.2 & 0.3 & 0.3 & 0.1 \\ 3 & 0.3 & 0.3 & 0.3 & 0.1 & 0 \\ 4 & 0.1 & 0.2 & 0.3 & 0.3 & 0.1 \end{array}$$

Le us explain why this is the case. First note that there is no back ordering and so the are no negative states in the chain. If the chain is in state 4, then there will be no ordering of new units and the transition to other states depend only on the demand ξ_n, thus, $P_{4,i} = Pr\{\xi = 4 - i\}$, i.e., the probability of reducing the amount of available stock, 4, by ξ units. Note that this distribution is identical to that of state 2 because in that state we order 2 additional units, which will be available at the beginning of the next period, and so we have 4 units total available at the beginning of the next period. Transition then depends on demand ξ as before.

For state 1, we order 2 additional units. So at the beginning of the next period we have 3 available units and so there is no way reach a stock of 4. Hence, there is no transition from state 1 to state 4. We can transition to other states as follow: from state 1 to state 2 if we get $\xi = 1$ unit of demand, from state 1 to state 0 if we get $\xi = 3$ or $\xi = 4$ (fulfilling as much demand as possible) units of demand. Finally, the probability of transition from state 1 to state 3 correspond to the case where there is no demand, i.e., $\xi = 0$ so we stay with our new inventory level of 3. Note that state 3 has the same distribution as state 1 since at state 3 we order no new units and so the distribution depends on demand as previously explained.

The final case is state 0 where we order two new units to reach a level of 2 units at the beginning of the next period. Clearly there is no transition to state 3 or 4. We can transition to state 0 if we get a demand of $2, 3$ or 4 units. Also, we can transition to state 1 if we get a demand of exactly 1 unit. Lastly, we can transition to state 2 only in case there is no demand, i.e., $\xi = 0$ so we stay with our new inventory level of 2.

Limiting distribution. Note that this is a regular M.C. We can reach any state from any other state and there is at least one positive probability of going back to the same state, say $P_{0,0} = 0.6 > 0$. We can find the limiting distribution using Theorem 1.1. and solving the linear system:

$$\pi P = \pi, \text{ and } \sum_{i=0}^{4} \pi_i = 1, \text{ from which we get the equations :}$$

$$\begin{array}{ll}
\frac{6}{10}\pi_0 + \frac{3}{10}\pi_1 + \frac{1}{10}\pi_2 + \frac{3}{10}\pi_3 + \frac{1}{10}\pi_4 = \pi_0 & 3\pi_1 + \pi_2 + 3\pi_3 + \pi_4 = 4\pi_0 \\
\frac{3}{10}\pi_0 + \frac{3}{10}\pi_1 + \frac{2}{10}\pi_2 + \frac{3}{10}\pi_3 + \frac{2}{10}\pi_4 = \pi_1 & 3\pi_0 + 2\pi_2 + 3\pi_3 + 2\pi_4 = 7\pi_1 \\
\frac{1}{10}\pi_0 + \frac{3}{10}\pi_1 + \frac{3}{10}\pi_2 + \frac{3}{10}\pi_3 + \frac{3}{10}\pi_4 = \pi_2 \quad \Longrightarrow & \pi_0 + 3\pi_1 + 3\pi_3 + 3\pi_4 = 7\pi_2 \\
\frac{1}{10}\pi_1 + \frac{3}{10}\pi_2 + \frac{1}{10}\pi_3 + \frac{3}{10}\pi_4 = \pi_3 & \pi_1 + 3\pi_2 + 3\pi_4 = 9\pi_3 \\
\frac{1}{10}\pi_2 + \frac{1}{10}\pi_4 = \pi_4 & \boxed{\pi_2 = 9\pi_4}
\end{array}$$

Let us write each π_i in terms of π_4. For this purpose we do not need all 5 equations, so we will ignore the first one. Replacing equation 5 into equation 4: $\pi_1 + 3(9\pi_4) + 3\pi_4 = 9\pi_3 \Longrightarrow \pi_1 = 9\pi_3 - 30\pi_4$. Replacing this expression into equation 3: $\pi_0 + 30\pi_3 - 87\pi_4 = 63\pi_4 \Longrightarrow \pi_0 = 150\pi_4 - 30\pi_3$. Replacing into equation 2: $3(150\pi_4 - 30\pi_3) + 2(9\pi_4) + 3\pi_3 + 2\pi_4 = 7(9\pi_3 - 30\pi_4) \Longrightarrow \pi_3 = \frac{68}{15}\pi_4$. Now we can solve for π_1 in terms of π_4, i.e., $\pi_1 = 9(\frac{68}{15}\pi_4) - 30\pi_4 \Longrightarrow \pi_1 = \frac{162}{15}\pi_4$. Lastly, solve for π_0 in terms of π_4 to get: $\pi_0 = 150\pi_4 - 30(\frac{68}{15}\pi_4) \Longrightarrow \pi_0 = 14\pi_4$. Now we can use the final condition to solve for the value of π_4:

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \Longrightarrow \pi_4(14 + \frac{162}{15} + 9 + \frac{68}{15} + 1) = 1 \Longrightarrow \pi_4(\frac{210 + 162 + 135 + 68 + 15}{15}) = 1 \Longrightarrow \pi_4 = \frac{15}{590}$$

Solving for each π_i, we obtain the limiting distribution:

$$(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = \left(\frac{210}{590}, \frac{162}{590}, \frac{135}{590}, \frac{68}{590}, \frac{15}{590}\right)$$

(c) In the long run, during what fraction of periods are orders placed?

Solution: Orders for inventory restock are placed if stock level is $0, 1$ or 2. Hence, in the long run, the fraction of periods orders are placed corresponds to:

$$\pi_0 + \pi_1 + \pi_2 = \frac{210}{590} + \frac{162}{590} + \frac{135}{590} = \frac{507}{590} = \boxed{0.85932203}$$

(2.5) Suppose that the weather on any day depends on the weather conditions during the previous two days. We form a Markov chain with the following states:

State (S, S) if it was sunny both today and yesterday,

State (S, C) if it was sunny yesterday but cloudy today,

State (C, S) if it was cloudy yesterday but sunny today,

State (C, C) if it was cloudy both today and yesterday,

and transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ (S,S) \\ (S,C) \\ (C,S) \\ (C,C) \end{array} \begin{array}{c} \begin{array}{cccc} (S,S) & (S,C) & (C,S) & (C,C) \end{array} \\ \left\| \begin{array}{cccc} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{array} \right\| \end{array}$$

(a) Given that it is sunny on days 0 and 1, what is the probability it is sunny on day 5?

Solution: Let $Y_n = S$ or C depending on whether day n was sunny or cloudy. We want to find:

$$Pr\{Y_5 = S|Y_0 = S, Y_1 = S\}$$

We can use the law of total probability and compute this event by considering the disjoint events $Y_4 = S$ and $Y_4 = C$

$$Pr\{Y_5 = S|Y_0 = S, Y_1 = S\} = Pr\{Y_5 = S, Y_4 = S|Y_0 = S, Y_1 = S\} + Pr\{Y_5 = S, Y_4 = C|Y_0 = S, Y_1 = S\}$$

Now we can write this probabilities in terms of our Markov chain: days 0 and 1 correspond to X_0 while, days 4 and 5 correspond to X_2. Note that days 2 and 3 correspond to state X_1.

$$\begin{aligned}
Pr\{Y_5 = S, Y_4 = S|Y_0 = S, Y_1 = S\} + Pr\{Y_5 = S, Y_4 = C|Y_0 = S, Y_1 = S\} &= Pr\{X_2 = (S,S)|X_0 = (S,S)\} \\
&+ Pr\{X_2 = (C,S)|X_0 = (S,S)\} \\
&= P^{(2)}_{(S,S),(S,S)} + P^{(2)}_{(S,S),(C,S)} \\
&= 0.7^2 + 0.3 \times 0.4 \\
&= \boxed{0.61}
\end{aligned}$$

(b) In the long run, what fraction of days are sunny?

Solution: Sunny days correspond to pairs (S,S) and (C,S), i.e., days were both the day before and the day after were sunny or days were the day before was cloudy but the next day was sunny. Since we want to find fraction of days, we need to compute the limiting distribution for the aforementioned cases. The fraction we are interested in is:

$$\pi_{(S,S)} + \pi_{(C,S)} = \text{ fraction of sunny days}$$

Let us compute the limiting distribution:

First note that this is a regular transition probability matrix. To see this, compute P^2 and verify that all entries are positive:

$$\mathbf{P^2} = \begin{array}{c} \\ (S,S) \\ (S,C) \\ (C,S) \\ (C,C) \end{array} \begin{array}{c} \begin{array}{cccc} (S,S) & (S,C) & (C,S) & (C,C) \end{array} \\ \left\| \begin{array}{cccc} 0.49 & 0.21 & 0.12 & 0.18 \\ 0.2 & 0.2 & 0.12 & 0.48 \\ 0.35 & 0.15 & 0.2 & 0.3 \\ 0.1 & 0.1 & 0.16 & 0.64 \end{array} \right\| \end{array}$$

We can find the limiting distribution using Theorem 1.1. and solving the linear system:

$\pi P = \pi$, and $\sum_{i=0}^{3} \pi_i = 1$, from which we get the equations (using the shorthand $0 = (S,S), 1 = (S,C), 2 = (C,S)$ and $3 = (C,C)$:

$$\begin{array}{l} \frac{7}{10}\pi_0 + \frac{5}{10}\pi_2 = \pi_0 \\ \frac{3}{10}\pi_0 + \frac{5}{10}\pi_2 = \pi_1 \\ \frac{4}{10}\pi_1 + \frac{2}{10}\pi_3 = \pi_2 \\ \frac{6}{10}\pi_1 + \frac{8}{10}\pi_3 = \pi_3 \end{array} \implies \begin{array}{l} 5\pi_2 = 3\pi_0 \\ 3\pi_0 + 5\pi_2 = 10\pi_1 \\ 4\pi_1 + 2\pi_3 = 10\pi_2 \\ 6\pi_1 = 2\pi_3 \end{array} \implies \quad \pi_2 = \tfrac{3}{5}\pi_0 \ (*)$$

Substituting (*) into the second equation we get: $3\pi_0 + 3\pi_0 = 10\pi_1 \implies \pi_1 = \frac{3}{5}\pi_0$. We can omit the third equation and use the fourth equation: $6(\frac{3}{5}\pi_0) = 2\pi_3 \implies \pi_3 = \frac{9}{5}\pi_0$. Finally, use the condition

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \implies \pi_0 + \frac{3}{5}\pi_0 + \frac{3}{5}\pi_0 + \frac{9}{5}\pi_0 = 1 \implies \pi_0(\frac{5+3+3+9}{5}) = 1 \implies \pi_0 = \frac{5}{20}$$

We can solve for the entire limiting distribution to get $(\pi_0, \pi_1, \pi_2, \pi_3) = (\frac{5}{20}, \frac{3}{20}, \frac{3}{20}, \frac{9}{20})$.

Hence,

$$\text{fraction of sunny days} = \pi_{(S,S)} + \pi_{(C,S)} = \pi_0 + \pi_2 = \frac{5}{20} + \frac{3}{20} = \frac{8}{20} = \boxed{\frac{2}{5}}$$

(2.6) Consider a computer system that fails on a given day with probability p and remains "up" with probability $q = 1 - p$. Suppose the repair time is a random variable N having the probability mass function $p(k) = \beta(1-\beta)^{k-1}$ for $k = 1, 2, \ldots,$ where $0 < \beta < 1$. Let $X_n = 1$ if the computer is operating on day n and $X_n = 0$ if not. Show that $\{X_n\}$ is a Markov chain with transition matrix

$$\mathbf{P} = \begin{array}{c|cc} & 0 & 1 \\ 0 & \alpha & \beta \\ 1 & p & q \end{array}$$

and $\alpha = 1 - \beta$. Determine the long run probability that the computer is operating in terms of $\alpha, \beta, p,$ and q.

Solution: Clearly, once in state 1 we can either go to state 0 with probability p or to state 1 with probability q. These are fixed numbers that have no dependency in the past or future. Now, in state 0 the computer system has fail and so it must be repaired. We can compute these probabilities:

$$Pr\{X_n = 1|X_{n-1} = 0\} = Pr\{\text{it takes one day to repair the computer}\} = Pr\{N = 1\} = \beta(1-\beta)^{1-1} = \beta$$

$$Pr\{X_n = 0|X_{n-1} = 0\} = Pr\{\text{it takes more than one day to repair the computer}\} = Pr\{N > 1\} = 1 - Pr\{N = 1\} = 1 - \beta = \alpha$$

Again, these probabilities only depend on one previous state. They are unaffected by past or future states. Since this is true for all states, we conclude that $\{X_n\}$ is a Markov chain.

Let us find the limiting distribution. Two cases:

a) Suppose $p > 0$. In this case the chain is regular: i) $P_{00} = \alpha = 1 - \beta > 0$ because $0 < \beta < 1$ and ii) There exists a path between 0 and 1 ($P_{0,1} = \beta > 0$) and between 1 and 0 ($P_{1,0} = p > 0$, by hypothesis).

b) Suppose $p = 0$. Then there is no stochasticity in the process, the computer system always work and there is no need to repair it. The long run distribution is $(\pi_0, \pi_1) = (0, 1)$, always in state 1.

For case a), we can find the limiting distribution using Theorem 1.1. and solving the linear system:

$$\pi P = \pi, \text{ and } \pi_0 + \pi_1 = 1, \text{ from which we get the equations:}$$

$$\begin{array}{lllll} \alpha\pi_0 + p\pi_1 = \pi_0 & \implies & p\pi_1 = (1-\alpha)\pi_0 & \implies & \pi_0 = \dfrac{p}{1-\alpha}\pi_1 \ (*) \\ \beta\pi_0 + q\pi_1 = \pi_1 & & \beta\pi_0 = (1-q)\pi_1 & & \end{array}$$

From (*) we can solve for π_1 as follows: $\pi_0 + \pi_1 = 1 \implies \dfrac{p}{1-\alpha}\pi_1 + \pi_1 = 1 \implies (\dfrac{p+1-\alpha}{1-\alpha})\pi_1 = 1 \implies \pi_1 = \dfrac{1-\alpha}{p+1-\alpha}$

For the sake of clarity, we can substitute $1 - \alpha = \beta$ in π_1 to get the limiting distribution:

$$\boxed{\pi_1 = \frac{\beta}{p+\beta}}, \quad \text{long run probability that the computer is operating}$$

This function makes sense, if $p >> \beta$, then π_1 is very small to account for the fact that the repair time is not keeping up with the requirements for repairing the system. On the contrary, if $p << \beta$, then π_1 is very large to account for the fact that the computer is being repaired very rapidly and so the "up" time increases in the long run.