M464 - Introduction To Probability II - Homework 6

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Chapter 4

(3.2) Show that a finite state aperiodic irreducible Markov chain is regular and recurrent.

Solution: Let us proof each case separately

Recurrent: Assuming the chain is finite, at least one state is going to be visited infinitely often starting from that state, since suppose this is not the case: then every state will be visited finitely many times starting from itself. Pick an arbitrary state i. Then $\sum_{n=1}^{\infty} P_{i,i}^{(n)} < \infty$. But if this is the case, then eventually none of the states will be visited. This is a contradiction because the chain is finite and so it has to go back to one state infinitely often. In other words, if all states are transient then eventually the chain is in none of the states which is absurd because the chain is finite. Therefore, at least one state is recurrent and by Corollary 3.1 and the hypothesis that the chain is irreducible, we can conclude that all states are recurrent, i.e., the chain is recurrent.

Regular: Having showed that the chain is recurrent we can apply Theorem 4.1. In the remark of the theorem we have that: "If $\lim_{n\to\infty} P_{i,i}^{(n)} > 0$ for one i in an aperiodic recurrent class, then $\pi_j > 0$ for all j in the class of i." Since we have only one class, then we can conclude that $\pi_j > 0$ for all states j. By definition, $\lim_{n\to\infty} P_{i,j}^{(n)} = \pi_j$, and thus eventually all states i, j will be accessible from each other in some number of transitions, i.e., there exists a number $K_{i,j}$ such that $P_{i,j}^{(k)} > 0$ for all $k > K_{i,j}$ (definition of limit). Finally, since we have a finite number of states, choose the maximum of all the $K_{i,j}$ such that the previous condition hold, call it L, and we will have that for all states i, j: $P_{i,j}^{(k)} > 0$ for all $k > L$, i.e., the chain is regular.

(3.3) Recall the first return distribution (Section 3.3)

$$f_{ii}^{(n)} = Pr\{X_1 \neq i, X_2 \neq i, \dots, X_{n-1} \neq i, X_n = i | X_0 = i\} \quad \text{for } n = 1, 2, \dots,$$

with $f_{ii}^{(0)} = 0$ by convention. Using equation (3.2), determine $f_{00}^{(n)}, n = 1, 2, 3, 4, 5$, for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right\| \end{array}$$

Solution: Equation (3.2) is $P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}$. To use this equation we are going to need up to the 5th power of P_{00}. A simple calculation shows that: $P_{00}^{(1)} = 0, P_{00}^{(2)} = \frac{1}{4}, P_{00}^{(3)} = \frac{1}{8}, P_{00}^{(4)} = \frac{3}{8}$ and $P_{00}^{(5)} = \frac{7}{32}$. Now, let us apply equation (3.2) in each case $n = 1, 2, 3, 4, 5$:

$$\begin{aligned} n = 1: \quad P_{00}^{(1)} &= \sum_{k=0}^{1} f_{00}^{(k)} P_{00}^{(1-k)} \\ &= f_{00}^{(0)} P_{00}^{(1)} + f_{00}^{(1)} P_{00}^{(0)} \\ &= f_{00}^{(1)} && \text{Since } f_{00}^{(0)} = 0 \text{ and } P_{00}^{(0)} = 1 \\ \implies & P_{00}^{(1)} = \boxed{f_{00}^{(1)} = 0} && \text{by transition matrix} \end{aligned}$$

$$
\begin{aligned}
n=2: \quad P_{00}^{(2)} &= \sum_{k=0}^{2} f_{00}^{(k)} P_{00}^{(2-k)} \\
&= f_{00}^{(0)} P_{00}^{(2)} + f_{00}^{(1)} P_{00}^{(1)} + f_{00}^{(2)} P_{00}^{(0)} \\
&= f_{00}^{(2)} && \text{Since } f_{00}^{(0)} = f_{00}^{(1)} = 0 \text{ and } P_{00}^{(0)} = 1 \\
\implies & P_{00}^{(2)} = \boxed{f_{00}^{(2)} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}} && \text{by transition matrix}
\end{aligned}
$$

$$
\begin{aligned}
n=3: \quad P_{00}^{(3)} &= \sum_{k=0}^{3} f_{00}^{(k)} P_{00}^{(3-k)} \\
&= f_{00}^{(0)} P_{00}^{(3)} + f_{00}^{(1)} P_{00}^{(2)} + f_{00}^{(2)} P_{00}^{(1)} + f_{00}^{(3)} P_{00}^{(0)} \\
&= \tfrac{1}{4} P_{00}^{(1)} + f_{00}^{(3)} && \text{Since } f_{00}^{(0)} = f_{00}^{(1)} = 0 \text{ and } P_{00}^{(0)} = 1 \\
\implies & f_{00}^{(3)} = P_{00}^{(3)} - \tfrac{1}{4} P_{00}^{(1)} = \tfrac{1}{8} - \tfrac{1}{4} \cdot 0 = \boxed{\frac{1}{8} = f_{00}^{(3)}}
\end{aligned}
$$

$$
\begin{aligned}
n=4: \quad P_{00}^{(4)} &= \sum_{k=0}^{4} f_{00}^{(k)} P_{00}^{(4-k)} \\
&= f_{00}^{(0)} P_{00}^{(4)} + f_{00}^{(1)} P_{00}^{(3)} + f_{00}^{(2)} P_{00}^{(2)} + f_{00}^{(3)} P_{00}^{(1)} + f_{00}^{(4)} P_{00}^{(0)} \\
&= \tfrac{1}{4} P_{00}^{(2)} + \tfrac{1}{8} P_{00}^{(1)} + f_{00}^{(4)} && \text{Since } f_{00}^{(0)} = f_{00}^{(1)} = 0 \text{ and } P_{00}^{(0)} = 1 \\
\implies & f_{00}^{(4)} = P_{00}^{(4)} - \tfrac{1}{16} = \tfrac{3}{8} - \tfrac{1}{16} = \boxed{\frac{5}{16} = f_{00}^{(4)}}
\end{aligned}
$$

$$
\begin{aligned}
n=5: \quad P_{00}^{(5)} &= \sum_{k=0}^{5} f_{00}^{(k)} P_{00}^{(5-k)} \\
&= f_{00}^{(0)} P_{00}^{(5)} + f_{00}^{(1)} P_{00}^{(4)} + f_{00}^{(2)} P_{00}^{(3)} + f_{00}^{(3)} P_{00}^{(2)} + f_{00}^{(4)} P_{00}^{(1)} + f_{00}^{(5)} P_{00}^{(0)} \\
&= \tfrac{1}{4} P_{00}^{(3)} + \tfrac{1}{8} P_{00}^{(2)} + \tfrac{5}{16} P_{00}^{(1)} + f_{00}^{(5)} && \text{Since } f_{00}^{(0)} = f_{00}^{(1)} = 0 \text{ and } P_{00}^{(0)} = 1 \\
\implies & f_{00}^{(5)} = \tfrac{7}{32} - \tfrac{1}{32} - \tfrac{1}{32} = \boxed{\frac{5}{32} = f_{00}^{(5)}}
\end{aligned}
$$

(4.1) Consider the Markov chain on $\{0, 1\}$ whose transition probability matrix is

$$
\begin{array}{cc}
 & \begin{array}{cc} 0 & 1 \end{array} \\
\mathbf{P} = \begin{array}{c} 0 \\ 1 \end{array} & \left\| \begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array} \right\|,
\end{array}
\quad 0 < \alpha, \beta < 1.
$$

(a) Verify that $(\pi_0, \pi_1) = (\beta/(\alpha+\beta), \alpha/(\alpha+\beta))$ is a stationary distribution.

Solution: Since this is a regular matrix we can find the limiting distribution using Theorem 1.1. and solving the linear system:

$\pi P = \pi,$ and $\pi_0 + \pi_1 = 1$, from which we get the equations :

$$
\begin{aligned}
\pi_0(1-\alpha) + \pi_1\beta = \pi_0 \quad &\implies \quad \pi_1\beta = \pi_0\alpha \quad \implies \quad \pi_0 = \frac{\beta}{\alpha}\pi_1 \\
\pi_0\alpha + \pi_1(1-\beta) = \pi_1 \quad & \qquad\quad\;\; \pi_0\alpha = \pi_1\beta
\end{aligned}
$$

Replacing the last equation into the equation $\pi_0 + \pi_1 = 1$, we get $\frac{\beta}{\alpha}\pi_1 + \pi_1 = 1 \Longrightarrow \pi_1(\frac{\beta}{\alpha} + 1) = 1 \Longrightarrow \boxed{\pi_1 = \frac{\alpha}{\alpha+\beta}}$.

Solving for π_0 we get: $\pi_0 = \frac{\beta}{\alpha}\frac{\alpha}{\alpha+\beta} \Longrightarrow \boxed{\pi_0 = \frac{\beta}{\alpha+\beta}}$, thus $(\pi_0, \pi_1) = (\beta/(\alpha+\beta), \alpha/(\alpha+\beta))$

(b) Show that the first return distribution to state 0 is given by $f_{00}^{(1)} = (1-\alpha)$ and $f_{00}^{(n)} = \alpha\beta(1-\beta)^{n-2}$ for $n = 2, 3, \ldots$

Solution: By definition, $f_{00}^{(1)} = Pr\{X_1 = 0 | X_0 = 0\} = P_{00} = (1-\alpha)$. Now, for an arbitrary $n \in \{2, 3, \ldots\}$,

$$\begin{aligned}
f_{00}^{(n)} &= Pr\{X_n = 0, X_{n-1} = 1, \ldots X_2 = 1, X_1 = 1 | X_0 = 0\}, \text{ By def. of first return} \\
&= Pr\{X_n = 0, X_{n-1} = 1, \ldots X_2 = 1, X_1 = 1, X_0 = 0\}, \text{ Conditional prob. and } Pr\{X_0 = 0\} = 1 \\
&= Pr\{X_n = 0 | X_{n-1} = 1, \ldots X_2 = 1, X_1 = 1, X_0 = 0\} Pr\{X_{n-1} = 1, \ldots X_2 = 1, X_1 = 1, X_0 = 0\}, \text{ Conditional prob} \\
&= Pr\{X_n = 0 | X_{n-1} = 1\} Pr\{X_{n-1} = 1 | X_{n-2} = 1\} Pr\{X_{n-2} = 1, \ldots X_2 = 1, X_1 = 1, X_0 = 0\}, \text{ Markov property} \\
&\vdots \\
&= Pr\{X_n = 0 | X_{n-1} = 1\} Pr\{X_{n-1} = 1 | X_{n-2} = 1\} \cdots Pr\{X_2 = 1 | X_1 = 1\} Pr\{X_1 = 1 | X_0 = 0\}, \text{ factoring all terms} \\
&= P_{10} P_{11} \cdots P_{11} P_{01}, \text{ by definition of transition probabilities} \\
&= \beta P_{11}^{n-2} \alpha \\
&= \beta(1-\beta)^{n-2}\alpha = \boxed{\alpha\beta(1-\beta)^{n-2}}
\end{aligned}$$

This answer makes intuitive sense: the probability of first return to state 0 in n steps beginning at state 0 is the probability of first going to state 1 and staying in this state for $n-2$ transitions and then transitioning from state 1 to state 0. Since this is a Markov chain, this reduces to the product of the corresponding transitions.

(c) Calculate the mean return time $m_0 = \sum_{n=1}^{\infty} n f_{00}^{(n)}$ and verify that $\pi_0 = 1/m_0$

Solution: This is a recurrent, irreducible aperiodic Markov chain. By theorem 4.1 we know:

$$\lim_{n\to\infty} P_{00}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{00}^{(n)}} = \frac{1}{m_0}$$

In part a) we computed $\lim_{n\to\infty} P_{00}^{(n)} = \beta/(\alpha+\beta)$, hence,

$$m_0 = \frac{1}{\pi_0} = \frac{1}{\beta/(\alpha+\beta)} = \boxed{\frac{\alpha+\beta}{\beta}}$$

Now, let us compute m_0 directly:

$$\begin{aligned}
m_0 &= \sum_{n=1}^{\infty} n f_{00}^{(n)} \\
&= (1-\alpha) + \sum_{n=2}^{\infty} n f_{00}^{(n)} && \text{starting sum at } n = 1 \\
&= (1-\alpha) + \sum_{n=2}^{\infty} n\alpha\beta(1-\beta)^{n-2} && \text{by part (b)} \\
&= (1-\alpha) + \alpha\beta \sum_{n=2}^{\infty} n(1-\beta)^{n-2} && \text{since } \alpha\beta \text{ is a constant}
\end{aligned}$$

We know that $|(1-\beta)| < 1$, and so we can use the following series expansion to compute the sum $\sum_{n=2}^{\infty} n(1-\beta)^{n-2}$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Longrightarrow \text{differentiate both sides: } \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

Make the change of variable $n-2=m-1 \Longrightarrow m=n-1$ for $\sum_{n=2}^{\infty} n(1-\beta)^{n-2} \Longrightarrow \sum_{m=1}^{\infty}(m+1)(1-\beta)^{m-1}$.
Now we can solve this sum using the series expansion:

$$\begin{aligned}
\sum_{m=1}^{\infty}(m+1)(1-\beta)^{m-1} &= \sum_{m=1}^{\infty} m(1-\beta)^{m-1}+\sum_{m=1}^{\infty}(1-\beta)^{m-1} && \text{separating sum} \\
&= \frac{1}{(1-(1-\beta))^2}+\frac{1}{(1-(1-\beta))} && \text{series expansion and geometric sum} \\
&= \frac{1}{\beta^2}+\frac{1}{\beta} \\
&= \frac{1+\beta}{\beta^2}
\end{aligned}$$

Replacing this into our equation for m_0 we obtain the desired result:

$$m_0=(1-\alpha)+\alpha\beta\sum_{n=2}^{\infty} n(1-\beta)^{n-2}=1-\alpha+\alpha\beta\left[\frac{1+\beta}{\beta^2}\right]=1-\alpha+\frac{\alpha+\alpha\beta}{\beta}=\frac{\beta-\alpha\beta+\alpha+\alpha\beta}{\beta}=\boxed{\frac{\alpha+\beta}{\beta}=\frac{1}{\pi_0}}$$

(4.4) Let $\{\alpha_i : i=1,2,\ldots\}$ be a probability distribution, and consider the Markov chain whose transition probability matrix is

$$\mathbf{P}=\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & \cdots \\
0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 0 & 1 & 0 & 0 & 0 & \cdots \\
3 & 0 & 0 & 1 & 0 & 0 & \cdots \\
4 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

What condition on the probability distribution $\{\alpha_i : i=1,2,\ldots\}$ is necessary and sufficient in order that a limiting distribution exist, and what is this limiting distribution? Assume $\alpha_1>0$ and $\alpha_2>0$, so that the chain is aperiodic.

Solution: Let us try to solve for the limiting distribution π_j, i.e., $\pi\mathbf{P}=\pi$ and $\sum_{i=0}^{\infty}\pi_i=1$:

$$\begin{aligned}
&\alpha_1\pi_0+\pi_1=\pi_0 \\
&\alpha_2\pi_0+\pi_2=\pi_1 \\
&\alpha_3\pi_0+\pi_3=\pi_2 \\
&\alpha_4\pi_0+\pi_4=\pi_3 \\
&\qquad\vdots \\
&\alpha_i\pi_0+\pi_i=\pi_{i-1}, \text{ for } i=1,2,3,\ldots
\end{aligned}$$

Ignoring the first equation and substituting all others into $\sum_{i=0}^{\infty}\pi_i=1$ we get:

$$\pi_0+\pi_1+\pi_2+\pi_3+\cdots=1 \Longrightarrow \pi_0+(\alpha_2\pi_0+\pi_2)+(\alpha_3\pi_0+\pi_3)+(\alpha_4\pi_0+\pi_4)+\cdots=1 \Longrightarrow \pi_0+\alpha_2\pi_0+\alpha_3\pi_0+\alpha_4\pi_0+\pi_2+\pi_3+\pi_4+\cdots=1$$

Again, substituting all equations but the first one we get:

$$\pi_0+\alpha_2\pi_0+\alpha_3\pi_0+\alpha_4\pi_0+\alpha_3\pi_0+\pi_3+\alpha_4\pi_0+\pi_4+\alpha_5\pi_0+\pi_5+\cdots=1 \Longrightarrow$$

$$\pi_0+\alpha_2\pi_0+2\alpha_3\pi_0+2\alpha_4\pi_0+\alpha_5\pi_0+\pi_3+\pi_4+\pi_5+\cdots=1$$

Let us do one more substitution step:

$$\pi_0+\alpha_2\pi_0+2\alpha_3\pi_0+2\alpha_4\pi_0+\alpha_5\pi_0+\alpha_4\pi_0+\pi_4+\alpha_5\pi_0+\pi_5+\cdots=1 \Longrightarrow$$

$$\pi_0+\alpha_2\pi_0+2\alpha_3\pi_0+3\alpha_4\pi_0+2\alpha_5\pi_0+\pi_4+\pi_5+\cdots=1$$

Continuing this process we get:

$$\pi_0(1+\alpha_2+2\alpha_3+3\alpha_4+4\alpha_5+\cdots)=1 \iff \pi_0=\frac{1}{1+\alpha_2+2\alpha_3+3\alpha_4+4\alpha_5+\cdots} \iff \pi_0=\frac{1}{1+\alpha_2+\sum\limits_{i=2}^{\infty} i\alpha_{i+1}}$$

This is a recurrent, irreducible and aperiodic Chain. It is sufficient for it to be positive recurrent for it to have a limiting distribution. This happens if $\pi_j > 0$ which in this case means that $\pi_0 = \dfrac{1}{1+\alpha_2+\sum\limits_{i=2}^{\infty} i\alpha_{i+1}} > 0 \iff \sum\limits_{i=2}^{\infty} i\alpha_{i+1} < \infty$. Note that since this is an irreducible Chain and being positive recurrent is a class property, it suffices to check the condition for π_0. Conversely, if $\sum\limits_{i=2}^{\infty} i\alpha_{i+1} < \infty \iff \pi_0 = \dfrac{1}{1+\alpha_2+\sum\limits_{i=2}^{\infty} i\alpha_{i+1}} > 0$, so the chain is positive recurrent, irreducible, and aperiodic, so it has a limiting distribution. The necessary and sufficient condition is :

$$\boxed{\sum_{i=2}^{\infty} i\alpha_{i+1} < \infty}$$

(4.6) Determine the period of state 0 in the Markov chain whose transition probability matrix is:

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \\ -4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 3 & 2 & 1 & 0 & -1 & -2 & -3 & -4 \end{array} \\ \left\| \begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right\| \end{array}$$

Solution: By definition the period of state 0 is: $d(0) = gcd\{n \geq 1; P_{00}^{(n)} > 0\}$. From the transition matrix we see that the following transitions are possible: $0 \to 1 \to 2 \to 3 \to 0$. Therefore, $P_{00}^{(4)} > 0$. Another possibility is $0 \to -1 \to -2 \to -3 \to -4 \to 0$, which means that $P_{00}^{(5)} > 0$. Hence, $d(0) = gcd\{n \geq 1; P_{00}^{(n)} > 0\} = \{4, 5, \ldots\} = \boxed{1 = d(0)}$ since 4 and 5 are relatively prime. Note that this is an irreducible matrix and thus, all states have the same period.

(4.7) An individual either drives his car or walks in going from his home to his office in the morning, and from his office to his home in the afternoon. He uses the following strategy: If it is raining in the morning, then he drives the car, provided it is at home to be taken. Similarly, if it is raining in the afternoon and his car is at the office, then he drives the car home. He walks on any morning or afternoon that it is not raining or the car is not where he is. Assume that, independent of the past, it rains during successive mornings and afternoons with constant probability p. In the long run, on what fraction of *days* does our man walk in the rain? What if he owns two cars?

Solution: We can model this situation with a two state Markov chain with states C = the car is where the man is and NC = the car is not where the man is. The transition probability matrix is given by:

$$\mathbf{P} = \begin{array}{c} \\ C \\ NC \end{array} \begin{array}{c} \begin{array}{cc} C & NC \end{array} \\ \left\| \begin{array}{cc} p & 1-p \\ 1 & 0 \end{array} \right\| \end{array}$$

If it rains then the man takes his car. This happens with probability p. If it doesn't rain, then the man will walk and leave the car behind. This happens with probability $1-p$. Now, the fraction of days he walks in the rain is given by the event that he doesn't have the car and it rains OR he has the car and it doesn't rain but then it rains on the way back. The fraction of time he has or doesn't have the car is giving by the limiting distribution of the Markov chain (assuming $p > 0$, this is a regular matrix so we can use theorem 1.1):

$$\pi P = \pi, \text{ and } \pi_C + \pi_{NC} = 1, \text{ from which we get the equations:}$$

$$\begin{aligned} p\pi_C + \pi_{NC} &= \pi_C \\ (1-p)\pi_C &= \pi_{NC} \end{aligned}$$

Substituting the last equation into $\pi_C + \pi_{NC} = 1$ we get $\pi_C + (1-p)\pi_C = 1 \Longrightarrow \pi_C(1+1-p) = 1 \Longrightarrow \boxed{\pi_C = \frac{1}{2-p}}$.
This is the fraction of time the car is where he is. Note that this equation makes sense: if $p = 1$ the $\pi_C = 1$, he always has the car with him (he always need it because it is always raining). If $p = 0$ then $\pi_C = 1/2$, the car is always at home and he moves equally between home and work. From this we can compute $\pi_{NC} = (1-p)\pi_C \Longrightarrow \boxed{\pi_{NC} = \frac{1-p}{2-p}}$. Now we can compute the fraction we are interested in:

$$\begin{aligned}
\text{Fraction walk in the rain} &= Pr\{\text{doesn't have car and rains OR has car and it doesn't rain but then it rains on the way back}\} \\
&= \pi_{NC} Pr\{\text{Rains}\} + \pi_C Pr\{\text{Doesn't rain}\} Pr\{\text{Rains}\} \\
&= \pi_{NC} \cdot p + \pi_C \cdot (1-p) \cdot p \\
&= \frac{1-p}{2-p} \cdot p + \frac{1}{2-p} \cdot (1-p) \cdot p \\
&= \boxed{\frac{2(1-p)p}{2-p}}
\end{aligned}$$

The fraction of days he walks in the rain correspond to the cases where it rains and he doesn't have the car. In a given day this could happen in two ways: either he doesn't have the car in the morning and it rains or, he has the car in the morning and it doesn't rain, but it rains on the way back.

Now, let us consider the case where there are 2 cars. We can model this situation with a three state Markov chain with states $0, 1$ and 2, where each state represents the number of cars that are where the man is. The transition probability matrix is given by:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{array} \right\| \end{array}$$

If there are no cars where he is, then the two cars are at the other location. If there are two cars where he is then he might take one car, provided it rains, and there will be two cars in one location. If it doesn't rain, then there will be one car in each location. Finally, if there is only one car where he is, then the other car is at the other location and will remain this way provided it doesn't rain. If it rains, then he will take one of the cars to the other location and both cars will be in one location.

Just like before, assuming $p > 0$, we can compute the limiting distribution:

$$\pi P = \pi, \text{ and } \pi_0 + \pi_1 + \pi_2 = 1, \text{ from which we get the equations:}$$

$$\begin{aligned}
(1-p)\pi_2 &= \pi_0 \\
(1-p)\pi_1 + p\pi_2 &= \pi_1 \\
\pi_0 + p\pi_1 &= \pi_2
\end{aligned}$$

Substitute the second into the third eq.: $(1-p)\pi_2 + p\pi_1 = \pi_2 \Longrightarrow p\pi_1 = \pi_2 + (p-1)\pi_2 = \pi_2(p-1+1) = p\pi_2 \Longrightarrow \boxed{\pi_1 = \pi_2}$
Substitute this into third equation: $\pi_0 + p\pi_2 = \pi_2 \Longrightarrow \boxed{\pi_0 = (1-p)\pi_2}$

Substitue boxed equations into: $\pi_0 + \pi_1 + \pi_2 = 1 \Longrightarrow (1-p)\pi_2 + \pi_2 + \pi_2 = 1 \Longrightarrow \pi_2(1-p+1+1) = 1 \Longrightarrow \boxed{\pi_2 = \pi_1 = \frac{1}{3-p}}$

Thus, $\boxed{\pi_0 = \frac{1-p}{3-p}}$. Like before, the fraction of time he is where there are no cars is π_0. Likewise, the fraction of time he is where there are 1 or 2 cars is π_1, which is the same as π_2. Now we can compute the fraction we are interested in:

$$\begin{aligned}
\text{Fraction walk in the rain} &= Pr\{\text{no cars where he is and rains OR 2 cars where he is and doesn't rain but rains on way back}\} \\
&= \pi_0 Pr\{\text{Rains}\} + \pi_2 Pr\{\text{Doesn't rain}\} Pr\{\text{Rains}\} \\
&= \pi_0 \cdot p + \pi_2 \cdot (1-p) \cdot p \\
&= \frac{1-p}{3-p} \cdot p + \frac{1}{3-p} \cdot (1-p) \cdot p \\
&= \boxed{\frac{2(1-p)p}{3-p}}
\end{aligned}$$