M464 - Introduction To Probability II - Homework 8

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Chapter 5

2.4 Suppose that N points are uniformly distributed over the interval $[0, N)$. Determine the probability distribution for the number of points in the interval $[0, 1)$ as $N \to \infty$.

Solution:
Let $X_1, X_2, \ldots, X_N$ be N independent random variables with distribution $X_i \sim Uniform([0, N))$, for all i. Let,

$$Y_i = \begin{cases} 1 & \text{if } X_i \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$

Note that $Pr\{Y_i = 1\} = Pr\{X_i \in [0,1)\} = \dfrac{1-0}{N-0} = \dfrac{1}{N}$, by the uniform cdf of uniform distribution on $[0, N)$.
Thus, $Y_i \sim Bernoulli(p = \dfrac{1}{N})$. Now, let Z = number of points in the interval $[0, 1)$. Then, $Z = \sum_{i=1}^{N} Y_i \sim Binomial(N, \dfrac{1}{N})$.
By fact given in class we know that $Bin(m, \dfrac{\mu}{m}) \implies Pois(\mu)$ as $m \to \infty$. Applying this result we get:

$$Z = \sum_{i=1}^{N} Y_i \sim Binomial(N, \frac{1}{N}) \implies Z \sim Pois(1) \text{ as } N \to \infty$$

Hence, the probability distribution for the number of points in the interval $[0, 1)$ is $Pois(1)$ as $N \to \infty$.

2.7 N bacteria are spread independently with uniform distribution on a microscope slide of area A. An arbitrary region having area a is selected for observation. Determine the probability of k bacteria within the region of area a. Show that as $N \to \infty$ and $a \to 0$ such that $(a/A)N \to c$ $(0 < c < \infty)$, then $p(k) \to e^{-c}c^k/k!$.

Solution: Fix an arbitrary region of area a. The proportion of bacteria in this area is $\dfrac{a}{A}$. Let,

$$Y_i = \begin{cases} 1 & \text{if bacteria } i \text{ is in the region } a, \text{ for } i = 1, 2, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

Note that since bacteria is uniformly distributed over the entire region A, we have that $Pr\{Y_i = 1\} = \dfrac{a}{A}$, i.e., the chance of a bacteria being in the area a is equal to the proportion of bacteria in such area. Thus, $Y_i \sim Bernoulli(p = \dfrac{a}{A})$. Then,

$$Z = \text{number of bacteria in region } a = \sum_{i=1}^{N} Y_i \sim Binomial(N, \frac{a}{A})$$

So, the probability of k bacteria within the region of area a is $Pr\{Z = k\} = \binom{N}{k}\left(\dfrac{a}{A}\right)^k\left(1 - \dfrac{a}{A}\right)^{N-k}$, by Binomial pmf.
Now, let $N \to \infty$ and $a \to 0$ such that $(a/A)N \to c$ $(0 < c < \infty)$. Then:

$$\begin{aligned}
\binom{N}{k}\left(\frac{a}{A}\right)^k\left(1-\frac{a}{A}\right)^{N-k} &= \frac{N!}{(N-k)!k!}\left(\frac{a}{A}\right)^k\left(1-\frac{a}{A}\right)^{N-k} && \text{def. of binomial coefficient} \\
&= \frac{N(N-1)(N-2)\cdots(N-k+1)}{k!}\left(\frac{a}{A}\right)^k\left(1-\frac{a}{A}\right)^{N-k} && \text{def. of factorial} \\
&= \frac{N(N-1)(N-2)\cdots(N-k+1)}{k!}\left(\frac{c}{N}\right)^k\left(1-\frac{c}{N}\right)^{N-k} && \text{hypothesis} \\
&= \frac{N(N-1)(N-2)\cdots(N-k+1)}{N^k}\frac{c^k}{k!}\left(1-\frac{c}{N}\right)^{N-k} && \text{pulling } N^k \text{ out}
\end{aligned}$$

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\begin{aligned}
&= \frac{N}{N}\frac{N-1}{N}\frac{N-2}{N}\cdots\frac{N-k+1}{N}\frac{c^k}{k!}\left(1-\frac{c}{N}\right)^{N-k} && \text{rearranging terms} \\
&= 1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{k+1}{N}\right)\frac{c^k}{k!}\left(1-\frac{c}{N}\right)^{N-k} && \text{simplifying}
\end{aligned}
$$

Now we can analyze each piece:

$$1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{k+1}{N}\right) \to 1 \text{ as } N\to\infty,\ \text{because } \frac{i}{N} \to 0 \text{ as } N\to\infty$$

$$\left(1-\frac{c}{N}\right)^{N-k} = \frac{\left(1-\frac{c}{N}\right)^{N}}{\left(1-\frac{c}{N}\right)^{k}} \to \frac{e^{-c}}{1} = e^{-c} \text{ as } N\to\infty,\ \text{by limit definition of constant } e \text{ and } \frac{c}{N} \to 0 \text{ as } N\to\infty$$

$$\frac{c^k}{k!} \text{ are constants and so remain unchanged by taking limits}$$

Thus,

$$p(k) = \binom{N}{k}\left(\frac{a}{A}\right)^k\left(1-\frac{a}{A}\right)^{N-k} = 1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{k+1}{N}\right)\frac{c^k}{k!}\left(1-\frac{c}{N}\right)^{N-k} \to \frac{e^{-c}c^k}{k!}$$

$$\text{as } N\to\infty \text{ and } a\to 0 \text{ such that } (a/A)N \to c\ (0<c<\infty).$$

Note that this is Poisson with parameter c.