Enrique Areyon - M464 - Spring 2014 - Exam II

①

1. (a) $X(1) \sim Pois(4) \Rightarrow Pr\{X(1)=2\} = (e^{-4}\, 4^2)/2! = \boxed{8e^{-4}}$ ✓

(b) $X(1) \sim Pois(4)$ and $X(3) \sim Pois(12)$. Now,

$Pr\{X(1)=2 \text{ and } X(3)=6\} = \dfrac{Pr\{X(3)=6 \mid X(1)=2\}}{Pr\{X(1)=2\}}$ By conditional prob

Also, $Pr\{X(3)=6 \mid X(1)=2\} = Pr\{X(3)-X(1) = \cancel{6}\ 4\}$ By independence of increments

$X(3)-X(1) \sim Pois(2\cdot 4) = Pois(8)$; since sum of Pois. is Pois. Then,

$Pr\{X(1)=2 \text{ and } X(3)=6\} = \dfrac{Pr\{X(3)-X(1)=6\}}{Pr\{X(1)=2\}} = \dfrac{(e^{-8}\, 8^6)/6!}{(e^{-4}\, 4^2)/2!} = \dfrac{2!\, e^{-8}\, 8^6}{6!\, e^{-4}\, 4^2}$

$= \dfrac{e^{-4}\, 4^6 \cdot 2^7}{6!\, 4^2} = \boxed{\dfrac{e^{-4}\, 2^7}{6!}}$ ✗

(c) $P[X(1)=2 \mid X(3)=6] = \dfrac{P[X(1)=2 \text{ and } X(3)=6]}{P[X(3)=6]}$ By conditional prob

$= \dfrac{(e^{-4} \cdot 2^7)/6!}{(e^{-12} \cdot 12^6)/6!} = \boxed{\dfrac{e^{8} \cdot 2^7}{12^6}}$ ✗ OK

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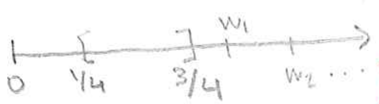
(d) $E[X(2)]$: since $X(2) \sim Pois(8) \Rightarrow E[X(2)] = 8$; rate of a poisson. ✓

4. (a)  ✗ So the probability of no arrival is the same as having the first arrival be after 3/4. We

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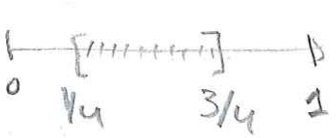
know the distribution of the first arrival: $W_1 \sim Exp(5)$. Hence, the prob. we want is $Pr\{W_1 > \frac{3}{4}\} = 1 - Pr\{W_1 \le \frac{3}{4}\} = 1 - (1 - e^{-5(\frac{3}{4})}) = \boxed{e^{-15/4}}$ ✗

(b) If we know $X(1)=n$, then $U_1, .., U_n \sim Uniform((0,1))$ are n independent uniform r.v.s representing the W's but with no order. We wish to compute

$Pr\{\text{no arrival in } [1/4, 3/4]\}$ ✗ $1 - Pr\{\text{all arrivals in } [1/4, 3/4]\}$

conditional

$= 1 - Pr\{\frac{1}{4} \le U_1 \le \frac{3}{4}, \frac{1}{4} \le U_2 \le \frac{3}{4}, \ldots, \frac{1}{4} \le U_n \le \frac{3}{4}\}$



$= 1 - [Pr\{\frac{1}{4} \le U_1 \le \frac{3}{4}\}]^n$ by independence and identical distributions

$= 1 - [\frac{1}{2}]^n$

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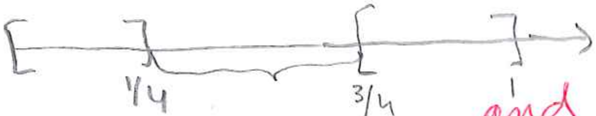
$= 1 - \frac{1}{2^n} = \boxed{\dfrac{2^n - 1}{2^n}}$ ✗

Does this make sense if $n=0$?

c) 

Again, Let $U_1, \dots, U_n \sim Uniform((0,1))$.

$Pr\{$no arrival in $[0, 1/4]$ ~~OR~~ and no arrival $[3/4]\} =$

$1 - Pr\{$all arrivals in $(1/4, 3/4)\} = 1 - Pr\{1/4 \le U_1 \le \frac{3}{4}, \dots, 1/4 \le U_n \le \frac{3}{4}\}$

$= 1 - \left[Pr\{1/4 \le U_1 \le 3/4\}\right]^n = 1 - \left[\frac{1}{2}\right]^n = \boxed{\frac{2^n - 1}{2^n}}$ ✗ 4

d) $\min\{W_1, 1 - W_n\} \mid X(1) = n$ we wish to compute 

$Pr\{\min\{W_1, 1 - W_n\} \le t \mid X(1) = n\}$

Let $U_1, \dots, U_n \sim Uniform((0,1))$

$\min\{W_1, 1 - W_n\} = \min\{\min\{U_1, \dots, U_n\}, 1 - \max\{U_1, \dots, U_n\}\}$

thus, $Pr\{\min\{W_1, 1 - W_n\} \underline{= t} \mid X(1) = n\}$ If $= t$, the $Pr = 0$.

$= Pr\{\underline{\min\{\min\{U_1, \dots, U_n\}, 1 - \max\{U_1, \dots, U_n\}} \mid X(1) = n\}$;

event?

But $0 < \max\{U_1, \dots, U_n\} \le 1 \Rightarrow$ max $< 0 \le 1 - \max\{U_1, \dots, U_n\} \Rightarrow \max\{U_1, \dots, U_n\} \le 1$

which means that $\min\{\min\{U_1, \dots, U_n\}, 1 - \max\{U_1, \dots, U_n\}\}$ ✗ $\min\{U_1, \dots, U_n\}$.

$Pr\{\min\{W_1, 1 - W_n\} > 1/4 \mid X(1) = n\} = Pr\{\min\{U_1, \dots, U_n\} > \frac{1}{4}\}$

3 $= \left(Pr\{U_1 > \frac{1}{4}\}\right]^n = \left[\frac{3}{4}\right]^n$ ✗

e) By the reasoning before

$Pr\{\min\{W_1, 1 - W_n\} > t\} = Pr\{U_1 > t\} = 1 - t$

$\Rightarrow Pr\{\min\{W_1, 1 - W_n\} \le t\} = 1 - (1 - t) = \boxed{t}$ ✗ This is not what you said in (d) for $t = \frac{1}{4}$.

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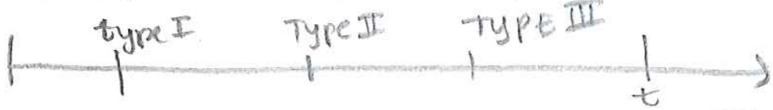
3) T = first time that at least one event has ocurred in each of the three processes.

Note: we can combine the process. Let $Y(t)$ be the combination of the three processes. then $Y(t)$ has rate $1^2+2^2+3^2=14$. We can think of each of this processes separetely by labels



the probability of getting an event of type I is $\frac{1}{4}$, of type II $\frac{4}{14}$, of type III is $\frac{9}{14}$. Now:

$Pr\{T \le t\} = Pr\{$at least one event of each type occurred up to time t$\}$

$= Pr\{$at least one event of Type I,
" " " " of type II,
" " " " of type III$\}$ we can separete these by independence

$= Pr\{X_1(t) \geq 1, X_2(t) \geq 1, X_3(t) \geq 1\}$

$= [1 - Pr\{X_1(t)=0\}][1-Pr\{X_2(t)=0\}][1-Pr\{X_3(t)=0\}]$

$= [1-e^{-1t}][1-e^{-4t}][1-e^{-9t}]$ ✓ 15

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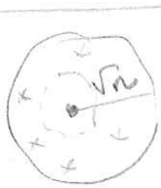
2) 

Let X = # of points within distance 1 of the center. Since points are distributed uniformly, the probability of having a point within distance 1 of the center should only depend on the proportion of the area.

So. Let $X_i = \begin{cases} 1 & \text{if point } i \text{ is in unit disk} \\ 0 & \text{otherwise.} \end{cases}$; $i=1,2,\dots,5n$.

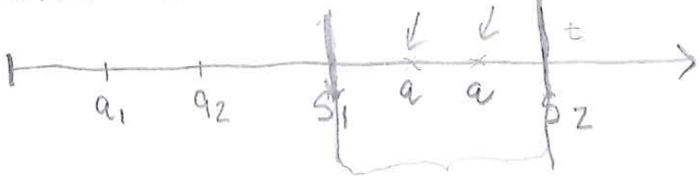
$Pr\{X_i = 1\} = \dfrac{\text{Area unit disk}}{\text{total area}} = \dfrac{\pi \cdot 1^2}{\pi(\sqrt{n})^2} = \dfrac{1}{n}$ ✓

Now, $X = \sum_{i=1}^{5n} X_i$. Since there is a small chance of each event occurring and there is a large area of ocurrence, we know that, since X is Binomial, the limiting distibution is poison, i.e., $X \sim Pois(5)$ ✓ 15

5) Let $X(t)$ = # of acorns fallen up to time t.
$S(t)$ = # of squirrels that pass up to time t.

$X(t) \sim Pois(1 \cdot t)$; $S(t) \sim Pois(2t)$; $X(t)$ indep. of $S(t)$

The situation can be diagram as follows:



Exactly two acorns between squirrel S1 and S2.

the second interarrival time for squirrels is $S_2 \sim Exp(2)$.

we want: $P_r\{X(S_2) - X(S_1) = 2\}$

$$= \int_0^\infty P_r\{X(S_2) - X(t) = 2\} f_{S_1}(t)\,dt$$; But $X(S_2) - X(t) \sim Pois((S_2 - t)t)$

By independence of incr.

$$= \int_0^\infty \frac{e^{-(S_2-t)t} ((S_2-t)t)^2}{2!} 2e^{-2t}\,dt$$? What is S_2?

Alternatively, we can just compute the probability of having two acorns up to time S1, by independence and identical distribution of the S_i; i.e.,
(plus memoryless property of exponential).

$$\int_0^\infty P_r\{X(t) = 2\} f_S(t)\,dt = \int_0^\infty \frac{e^{-t} t^2}{2!} 2e^{-2t}\,t\,dt$$

$$= \int_0^\infty e^{-3t} t^{\cancel{2}\,3}\,dt = P_r\{\text{second squirrel finds exactly Two acorns}\}.$$

$= ?$

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