Big-O Examples

Definition Let f and g be real-valued functions. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)| \quad \text{for all } x > k.$$

Example 1 Show that $f(x) = 4x^2 - 5x + 3$ is $O(x^2)$.

$$\begin{aligned}
|f(x)| &= |4x^2 - 5x + 3| \\
&\leq |4x^2| + |-5x| + |3| \\
&\leq 4x^2 + 5x + 3, \quad \text{for all } x > 0 \\
&\leq 4x^2 + 5x^2 + 3x^2, \quad \text{for all } x > 1 \\
&\leq 12x^2, \quad \text{for all } x > 1
\end{aligned}$$

We conclude that $f(x)$ is $O(x^2)$. Observe that $C = 12$ and $k = 1$ from the definition of big-O.

Example 2 Show that $f(x) = (x+5)\log_2(3x^2+7)$ is $O(x\log_2 x)$.

$$\begin{aligned}
|f(x)| &= |(x+5)\log_2(3x^2+7)| \\
&= (x+5)\log_2(3x^2+7), \quad \text{for all } x > -5 \\
&\leq (x+5x)\log_2(3x^2+7x^2), \quad \text{for all } x > 1 \\
&\leq 6x\log_2(10x^2), \quad \text{for all } x > 1 \\
&\leq 6x\log_2(x^3), \quad \text{for all } x > 10 \\
&\leq 18x\log_2 x, \quad \text{for all } x > 10
\end{aligned}$$

We conclude that $f(x)$ is $O(x\log_2 x)$. Observe that $C = 18$ and $k = 10$ from the definition of big-O.

Example 3 Show that $f(x) = (x^2 + 5\log_2 x)/(2x+1)$ is $O(x)$.

Since $\log_2 x < x$ for all $x > 0$, we conclude that

$$5\log_2 x < 5x < 5x^2, \quad \text{for all } x > 1.$$

Since $2x + 1 > 2x$, we conclude that

$$\frac{1}{2x+1} < \frac{1}{2x} \quad \text{for all } x > 0.$$

Therefore,

$$\begin{aligned}
|f(x)| &= \left|\frac{x^2 + 5\log_2 x}{2x+1}\right| \\
&= \frac{x^2 + 5\log_2 x}{2x+1}, \quad \text{for all } x > 1 \\
&\leq \frac{x^2 + 5x^2}{2x}, \quad \text{for all } x > 1 \\
&\leq 3x, \quad \text{for all } x > 1
\end{aligned}$$

We conclude that $f(x)$ is $O(x)$. Observe that $C = 3$ and $k = 1$ from the definition of big-O.