M301, Final Exam

December 20, 2013

Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page. Please show work when possible.

Name: ____

THE EXAM HAS A TOTAL OF 200 POINTS
NO CALCULATOR OR PHONE USE ALLOWED
DO NOT OPEN THE QUESTION BOOKLET UNTIL THE EXAM BEGINS

1. In parts (a) – (d), answer true or false. No justification is needed (only the true/false answer will be graded).

   (a) (5 points) If A is an $n \times n$ invertible matrix, then the columns of A must form a basis for $\mathbb{R}^n$.

   (b) (5 points) If A is an $n \times n$ matrix with n distinct eigenvalues, then A must be diagonalizable.

   (c) (5 points) If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$.

   (d) (5 points) If $H = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ must be a basis for H.

2. (a) (10 points) Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$. Express $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as a nontrivial linear combination of the columns of A.

(b) (10 points) For which vectors $\mathbf{b}$ does the system $A\mathbf{x} = \mathbf{b}$ have a unique solution? For which $\mathbf{b}$ are there infinitely many solutions? For which $\mathbf{b}$ are there no solutions?

3. (30 points) Find the nearest line, i.e. the least squares best linear fit, to the data points $(-1, 2)$, $(0, 7)$, $(1, -4)$, $(2, -1)$, and $(3, 1)$.
Please write your answer as the equation of a line $y = \beta_0 + \beta_1 x$.

4. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

(a) (10 points) Find an orthonormal basis for W.

(b) (5 points) If $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, find $\operatorname{proj}_W \mathbf{v}$.

5. (15 points) Let $A = \begin{bmatrix} 1.2 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$ and consider the dynamical system $A\mathbf{x}_k = \mathbf{x}_{k+1}$. Is the origin an attractor, a repeller, or a saddle point?

6. (20 points) Let $P = \begin{bmatrix} 1/5 & 1/2 \\ 4/5 & 1/2 \end{bmatrix}$ and consider the Markov chain $\mathbf{x}_{k+1} = P\mathbf{x}_k$.

If $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find a probability vector $\mathbf{q}$ such that $\mathbf{x}_k \to \mathbf{q}$ as $k \to \infty$.

7. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 5 \\ 1 & 1 & 1 \end{bmatrix}$, and let $B = A^{100}$ (that is, the 100th matrix power of A).

(a) (10 points) Find $\det(A)$.

(b) (10 points) Find $\det(B)$.

8. (a) (15 points) Diagonalize $\begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. You do not have to find P^{-1}.

(b) (5 points) There is usually more than one choice of P and D to diagonalize a given matrix. Does your P from part (a) satisfy $P^{-1} = P^T$? If not, is it possible to choose such a P? Either find one or explain why it is impossible.

9. (20 points) Let $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ and let the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find $\left[\mathbf{x}\right]_{\mathcal{B}}$.

10. (20 points) Find the value(s) of h such that the following vectors are linearly *dependent.*

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$