# Diagonalization
# Lay 5.3

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## 1 The importance of diagonal matrices

We talked last time about how easy it is to compute the action of a matrix on eigenvectors. It is even easier if the matrix is diagonal, since its eigenvectors are the standard basis. Notice that if $D$ is a diagonal matrix–for instance,

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix},$$

then

$$D \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Dx_1\mathbf{e}_1 + Dx_2\mathbf{e}_2 = 3x_1\mathbf{e}_1 + 7x_2\mathbf{e}_2 = \begin{bmatrix} 3x_1 \\ 7x_2 \end{bmatrix}.$$

It is also trivial to compute matrix powers:

$$D^2 = \begin{bmatrix} 9 & 0 \\ 0 & 49 \end{bmatrix}, \quad \text{etc.}$$

It is also easy to compute matrix powers if $A$ is similar to a diagonal matrix $D$:

**Example 1.1.** Let

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = PDP^{-1},$$

Where

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Then

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1} = PDP^{-1} = P \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} P^{-1}.$$

Similarly, for any positive integer $k$, $A^k = PD^kP^{-1}$.

## 2 Diagonalization

**Definition 2.1.** We say a matrix $A$ is **diagonalizable** if it is similar to a diagonal matrix $D$.

**Theorem 2.2.** *An $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors. In fact, $D$ is a diagonal matrix with $A = PDP^{-1}$ if and only if $P$ is a matrix whose columns are $n$ linearly independent eigenvectors of $A$. In this case, the $n$th diagonal entry in $D$ corresponds to the $n$th column of $P$.*

*Proof.* If $A = PDP^{-1}$ for a diagonal $D$, let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of $P$. Multiply on the right by $P$ to see that $AP = PD$. Now notice

$$AP = \begin{bmatrix} A\mathbf{v}_1 & \ldots & A\mathbf{v}_n \end{bmatrix}, \quad PD = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \ldots & \lambda_n\mathbf{v}_n \end{bmatrix}, \tag{1}$$

where $\lambda_i$ is the $i$th diagonal entry of $D$. Matching column by column, we see that the columns of $P$ are eigenvectors of $A$ and must be independent, since $P$ is invertible; moreover, $\lambda_i$ is the eigenvalue corresponding to $\mathbf{v}_i$. On the other hand, if $A$ has $n$ independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then set $P = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$. The columns of $P$ are independent, so it is invertible. Let us define $D = P^{-1}AP$, so that $A = PDP^{-1}$ and therefore $AP = PD$. By the same reasoning as the first equation in (1), we have $AP = \begin{bmatrix} A\mathbf{v}_1 & \ldots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \ldots & \lambda_n\mathbf{v}_n \end{bmatrix}$, where $\lambda_i$ is the $i$th eigenvalue of $A$. Therefore, $PD = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \ldots & \lambda_n\mathbf{v}_n \end{bmatrix}$. Since the columns of $PD$ are linear combinations of the columns of $P$, and since the columns of $P$ are linearly independent, it follows that $D$ must be diagonal. □

## 3 Diagonalization

We outline the diagonalization procedure for an $n \times n$ matrix $A$:

- Find the eigenvalues of $A$.
- Find bases for the corresponding eigenspaces.
- Figure out if you have $n$ linearly independent eigenvectors. If the sum of dimensions of the eigenspaces of $A$ is equal to $n$, you're set and the union of the bases for the different eigenspaces will consist of $n$ linearly independent eigenvectors. Otherwise, it's not diagonalizable.
- $P$ is the matrix whose columns are the eigenvectors.
- $D$ is the matrix whose $i$th entry is the eigenvalue for the $i$th column of $P$.

**Example 3.1.** Diagonalize

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

The eigenvalues of $A$ are $\lambda_1 = 4$ and $\lambda_2 = -2$. A basis for the $\lambda_1$ eigenspace is provided by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

and a basis for the $\lambda_2$ eigenspace is provided by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Therefore, the matrices

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

The eigenvectors we chose within each eigenspace were independent because we found a basis. Note that the eigenvectors from different eigenspaces are *automatically* independent, as predicted by our theorem from before.

It is not true that every matrix is diagonalizable. However, it is easy to see the following theorem holds:

**Theorem 3.2.** *If an $n \times n$ matrix has $n$ distinct eigenvalues, then it is diagonalizable.*

What about if there are not $n$ eigenvalues? Then the following theorem takes care of things:

**Theorem 3.3.** *If $A$ is $n \times n$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, then:*

- *For each $k$, the eigenspace corresponding to $\lambda_k$ has dimension $\leq$ the algebraic multiplicity of $\lambda_k$;*
- *$A$ is diagonalizable if and only if the sum of the dimensions of the eigenspaces is $n$ (we actually noted this above already);*
- *$A$ is diagonalizable if and only if (a) its characteristic polynomial factors into linear factors (i.e., if and only if it has $n$ real roots, possibly having multiplicity bigger than one) and (b) the dimension of the eigenspace corresponding to each eigenvalue $\lambda_k$ is equal to the algebraic multiplicity of $\lambda_k$;*
- *If $A$ is diagonalizable and if for each $k$, $\mathcal{B}_k$ is a basis for the eigenspace corresponding to $\lambda_k$, then the vectors in the union $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k$ form a basis for $\mathbb{R}^n$ of eigenvectors of $A$.*