Discrete Dynamical Systems
Lay 5.6

November 30, 2013

We have seen a bit of "dynamical systems" in the presentation of the Fibonacci numbers. Today we will talk more about them.

1 Dynamical Systems

A dynamical system is a sequence of vectors $\mathbf{x}_0, \mathbf{x}_1, \dots$ in $\mathbb{R}^n$ with an associated $n \times n$ matrix A, such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for all k. The idea, as in what we have seen before, is that $\mathbf{x}_k$ represents the state of some system at time k, and multiplying by A moves the system forward in time.

If A is diagonalizable, there is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ made up of eigenvectors of A with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. In particular, we can write

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$$

for some unique set of constants $c_1, \dots, c_n$. Applying A to both sides gives

$$\mathbf{x}_1 = \mathbf{A}x_0 = c_1\lambda_1\mathbf{v}_1 + \dots + c_n\lambda_n\mathbf{v}_n.$$

Note that the same λ_i may appear more than once if some eigenspace has dimension > 1. In general, applying A multiple times gives

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \dots + c_n\lambda_n^k\mathbf{v}_n.$$

So a dynamical system with a diagonalizable matrix A is easy to analyze.

Example 1.1. This is Lay's example, so we will go with it, though with some nicer numbers. Consider a forest populated by rats and owls. In the absence of owls, the rats will reproduce off of vegetable matter in the forest, but owls will kill the rats. On the other hand, the owls will starve to death without rats. The populations O_k, R_k of rats and owls in month k are given by the equations

$$O_{k+1} = \frac{1}{2}O_k + \frac{1}{4}R_k$$
$$R_{k+1} = -p\,O_k + \frac{3}{2}R_k.$$

Here p is a parameter that we will vary to see how greater predation affects the system.

Organizing the populations in the form of a vector $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, we see $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 1/2 & 1/4 \\ -p & 3/2 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda^2 - 2\lambda + \frac{p+3}{4}$, which has real roots if and only if $p \leq 1$. Moreover, in the case that $p < 1$, it is easy to see from the quadratic formula that there will be two distinct eigenvalues, and so A will be diagonalizable. If $p = 1$, there is only one eigenvalue, and a calculation of the eigenspace shows that A is not diagonalizable.

Let's choose $p = 1/9$. In this case, the eigenvalues are $\lambda_1 = 1 + \sqrt{2}/3$ and $\lambda_2 = 1 - \sqrt{2}/3$. Corresponding eigenvectors are $\mathbf{v}_1 = (3 - 2\sqrt{2}, 2/3)$ and $\mathbf{v}_2 = (3 + 2\sqrt{2}, 2/3)$.

Let's say we start with some population vector $\mathbf{x}_0$ with nonzero entries (we start with more than zero owls and rats). Writing $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, we see that

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2.$$

Now, notice that $0 < \lambda_2 < 1$ (in fact, $\lambda_2 \approx 0.057$). Therefore, $\lambda_2^k \to 0$ as $k \to \infty$ and so

$$\mathbf{x}_k \approx c_1\lambda_1^k\mathbf{v}_1$$

as $k \to \infty$. Now, $\lambda_1 \approx 1.94$ and so both the owls and rats grow in population. In fact, the population of each almost doubles month-by-month.

2 Trajectories

Given a dynamical system where $\mathbf{x}_{k+1} = A\mathbf{x}_k$, we call the sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ the system's **trajectory**, and think of it as a depiction of how the system evolves over time. In the case that A is diagonalizable, there is a lot we can say about how these trajectories look graphically.

It is best if you read Lay's subsections "Graphical Description of Solutions" and "Change of Variable" to get the whole picture here. The idea is that if an eigenvalue has absolute value > 1, it represents a direction or axis along which the trajectory gets pushed away from the origin, etc.