Ratio, Percentage, and Rates
Class Activity 7
Math-T101 Spring 2014

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Measurement without a unit is meaningless. However, one can compare different measurements without specifying a unit. For example, if a bookcase is three times as tall as another bookcase, then ratio of the height of the first to the height of the second is 3:1. It does not matter if we measure the height in inches, feet, or meters.

Ratio is a relationship between two quantities of the same kind. It represents how two quantities compare. The ratio between two quantities is $A : B$ if there is a unit such that the first quantity measures A units and the second measures B units. We can also have a ratio of more than two quantities, such as $A : B : C$ which basically follows the same definition.

Definition 1. *Two ratios are called equivalent, when one can be obtained from the other by multiplying all measurements by the same non-zero number.*

For example, if a recipe calls for 1/2 of a cup of flour and 1/4 of a cup of sugar, then the ratio of flour to sugar is (1/2):(1/4), or equivalently, 2:1, since there is twice as much flour as sugar. We can think of 2:1 as representing 2 one-quarter cups of flour and 1 one-quarter cup of sugar.

Problem 1. *An easy chocolate frosting recipe calls for 1 cup of sugar, 1/4 cup of butter, 1/4 cup of milk, and 3/4 cup of chocolate chips. Suppose that we only have 1/2 cup of chocolate chips, and want to make the frosting with the same ratio (and we have a large supply of other ingredients). How much of the other ingredients should we use to make as much frosting as possible?*

$\frac{3}{4}$ cup of chocolate chips $\xrightarrow{\div 3 \ \& \ \times 2}$ $\frac{1}{2} = \frac{2}{4}$ cup of chocolate chips

1 cup of sugar $\longrightarrow$ $\frac{2}{3}$ cup of sugar

$\frac{1}{4}$ cup of butter $\longrightarrow$ $\frac{2}{4.3} = \frac{1}{6}$ cup of sugor

$\frac{1}{4}$ cup of milk $\longrightarrow$ $\frac{1}{4} \div 3 \times 2 = \frac{1}{6}$ cup of milk

Note: One can also think of the above problem as a proportion, that is, we are looking for x and y such that $1{:}(1/4){:}(3/4) = x{:}y{:}(1/2)$. A proportion is a statement that two ratios are equal.

$\div 3 \ \& \ \times 2$

$x = 1 \div 3 \times 2 = \frac{1}{3} \times 2 = \frac{2}{3}$ | $y = \frac{1}{4} \div 3 \times 2 = \frac{1}{4} \cdot \frac{1}{3} \times 2 = \frac{2}{12} = \frac{1}{6}$

Algebraic Solution:

$8x$ = Susan, $4x$ = Linda, $5x$ = Tonya

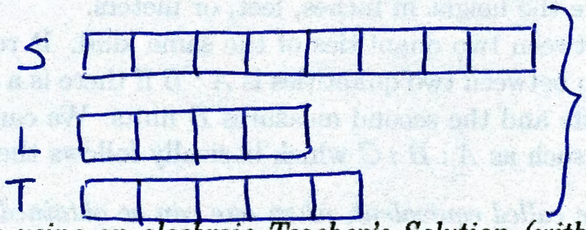
$S:L:T = 8:4:5$ Then $8x+4x+5x=187$ $\quad 8x=88$

$17x = 187$ $\quad 4x = 44$

$x = 11$ $\quad 5x = 55$

Problem 2. *The ratio of the number of Susan's marbles to Linda's is 2:1, and the ratio of Linda's marbles to Tanya's is 4:5. Find the ratio of Susan's marbles to Linda's to Tanya's. If the total number of marbles that they have is 187, how many marbles does each person have?*

$\left.\begin{array}{l} S:L = 2:1 = 8:4 \\ L:T = 4:5 \end{array}\right\} S:L:T = 8:4:5$



17 units = 187

1 unit = 11

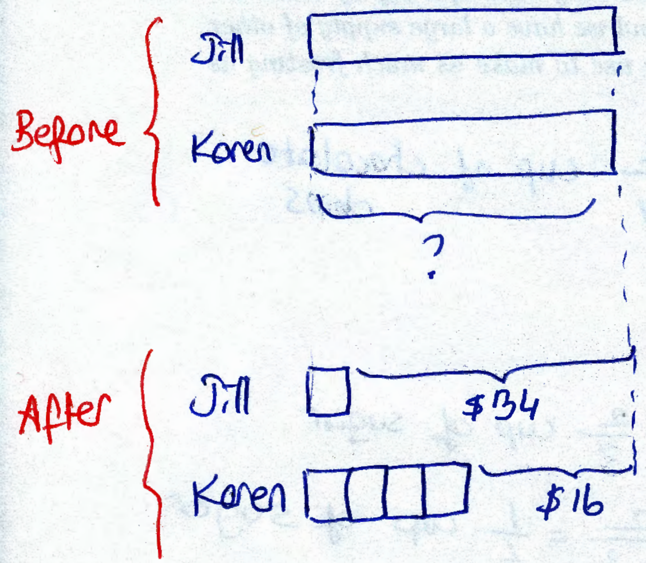
Susan → 88 marbles

Linda → 44 marbles

Tanya → 55 marbles

Problem 3. *Solve using an algebraic Teacher's Solution (with one variable) and illustrate with a bar diagram: Jill originally had the same amount of money as Karen. After Jill spends $34 and Karen spends $16, the ratio of Jill's money to Karen's is 1:4. How much did each have at first? (Optional exercise to try later: Give a Teacher's Solution without algebra, using a bar diagram. This is much harder.)*

Bar model Solution



$34 - 16 = 18 = 3$ units

$\$6 = 1$ unit

$34 + 6 = \$40$

Each had $40 at first

Algebraic Solution

Before: Jill's money = x, Karen's money = x

After: Jill's money = $x - 34$, Karen's money = $x - 16$

$1:4 = (x-34):(x-16)$

$\frac{1}{4} = \frac{x-34}{x-16}$

$x - 16 = 4\cdot(x-34)$

$x - 16 = 4x - 136$

$120 = 3x$

$40 = x$

Each had $40 at first

Problem 4. *The ingredients for a mud pie recipe are:*

- *6 ounces of chocolate Graham crackers*
- *7 tablespoons of butter*
- *1 quart of coffee ice cream, softened*
- *1/3 cup unsweetened cocoa powder*
- *2/3 cup granulated sugar*
- *1 1/3 cups heavy cream*
- *1 teaspoon vanilla extract*
- *1 ounce of semisweet chocolate*

Ben would like to prepare a mud pie from this recipe, but he does not have a scale available. He has a 100 gram bar of Lindt chocolate that is subdivided into 30 small rectangles. How many of these rectangles should he use to approximate one ounce? (You may use the fact that 100 grams are approximately 3.5 ounces.) He also has a box of Nabisco chocolate Graham crackers.The box contains 27 wafers that are subdivided into 4 crackers each. If the total weight of the contents of the box is 14.4 ounces, how many wafers should he use?

IMPORTANT: Ratios are not numbers. However, for the ease of calculations, sometimes we consider fractions related to the ratio of two numbers $A : B$. In this case, one should carefully SPECIFY the whole in the fraction, otherwise it will cause confusion and lead to mistakes. Examples of different fractions related to $A : B$ are:

- $\frac{A}{B}$ where the whole is the second quantity.
- $\frac{A}{A+B}$ where the whole is the total of the two quantities.
- $\frac{B}{A}$ where the whole is the first quantity.

Example 1. *The weights of a parcel containing clothes and a parcel containing books are in the ratio 4:7.*

1. *Express the weight of the clothes parcel as a fraction of the book parcel.*

$\frac{4}{7}$ = 4 parcels in relation to 7 parcels = 4 parcels out of 7 parcels

2. *Express the weight of clothes parcel as a fraction of the total weight of the two parcels.*

$\frac{4}{4+7} = \frac{4}{11}$ = 4 parcels of 11 total parcels

3. *Express the weight of the book parcel as a fraction of the clothes parcel.*

$\frac{7}{4}$ = 7 parcels of 4 parcels

Problem 5. *Discuss example 1.14 in the book p.172 in your group. Answer the two questions at the end of this example.*

$W : R = 6 : 10$

$W : R = (6+4) : (10+20)$

$W : R = 10 : 30 = 1 : 3$

→ So, Mary is right.

Diana added two ratios.

Ratios are not numbers, so we cannot add, subtract, divide, and multiply ratios.

Kevin wrote each ratio as fractions and then added them up. But ratios ~~[illegible]~~ can be interpretted as fractions only if they are describing the relationship between part and the whole.

This is not the case in here.

Algebraic Solution

Ratio → 3:7

Then $Sam = 3x$

$Lisa = 7x$

after →

$Sam = 3x - \frac{1}{6} \cdot 3x = 3x - \frac{x}{2} = \frac{5x}{2} = \frac{5}{2}x$

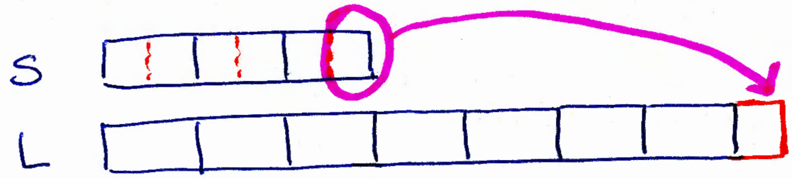
$Lisa = 7x + \frac{1}{6} \cdot 3x = 7x + \frac{x}{2} = \frac{15x}{2} = \frac{15}{2}x$

New Ratio

$\frac{5}{2} : \frac{15}{2} = 5:15 = 1:3$

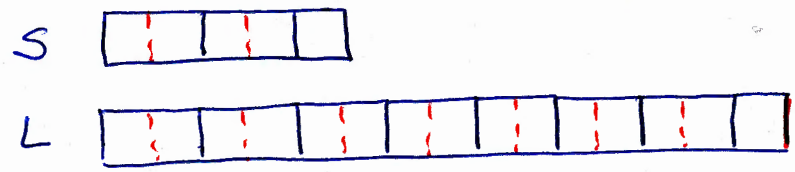
Problem 6. *Solve using an algebraic Teacher's Solution and illustrate with a bar diagram: Sam has $\frac{3}{7}$ as many stamps as Lisa. If Sam gives $\frac{1}{6}$ of his stamps to Lisa, what will be the ratio of Sam's to Lisa's?*

Before:



Ratio → 3:7 = 6:14

After:



Ratio → 5:15 = 1:3

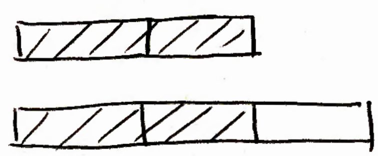
Problem 7. *In Mr. Nelson's class $\frac{2}{3}$ of the students are girls. In Ms. Cole's class $\frac{1}{3}$ of the students are girls. If Mr. Nelson and Ms. Cole combine their classes, will the ratio of boys to girls be 1:1?*

- If the class size is the same, then it is true.

Nelson's class

G

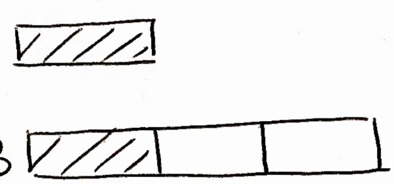
G+B



Cole's class

G

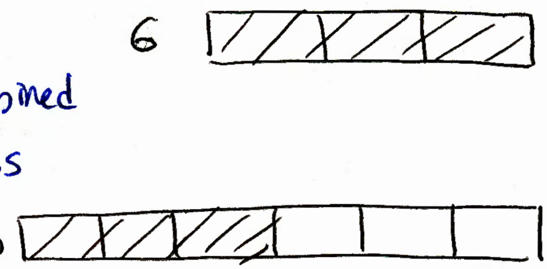
G+B



Combined class

G

G+B



3:6 → the ratio of girls to the whole class

3:3 = 1:1 → the ratio of girls to boys

- If the class sizes are not the same, then the ratio will be different.

$4x \quad 2x \quad 5x$

Problem 8. *Angela, Betty, and Carol shared a sum of money in the ratio 4:2:5.*

1. *What fraction of the sum of money did Carol receive?*

$$\frac{5x}{4x+2x+5x} = \frac{5x}{11x} = \frac{5}{11}$$

2. *If Carol received \$21 more than Betty, how much money did Angela receive?*

Algebraically

$5x - 2x = 3x$

$3x = \$21$

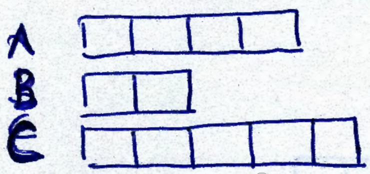
$x = \frac{21}{3} = 7$

Then $4x = 7 \cdot 4$

$4x = 28$

Angela received \$28.

Bar-model



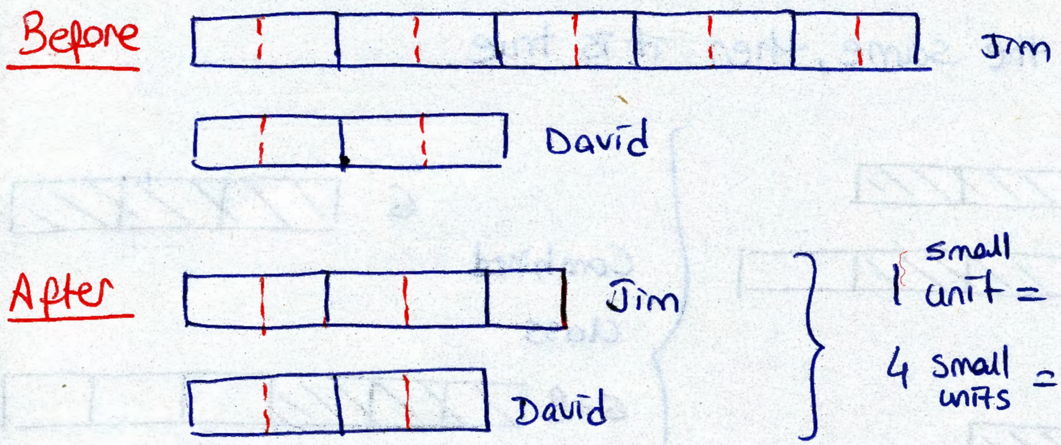
3 units = \$21

1 unit = \$7

4 units = \$28

Angela received \$28.

Problem 9. *The ratio of Jim's money to David's money was 5:2 at first. After Jim spent 1/2 of his money, and David did not spend any, Jim had \$20 dollars more than David. How much money did David have? How much money did Jim have at first?*



1 small unit = \$ 20

4 small units = \$ 80 → David had \$80

10 small units = \$200 → Jim had \$200

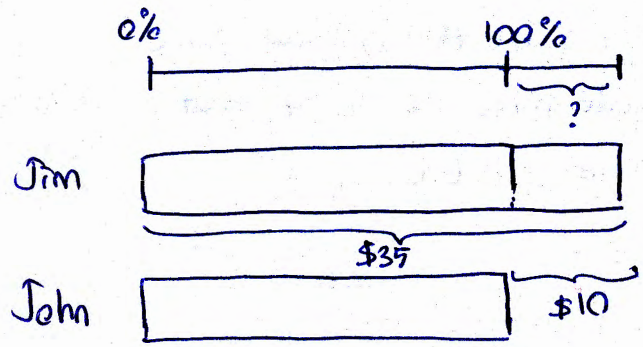
Question 1. *What is a percent?*

A percent is a specific type of fraction which has the denominator 100.

$\frac{1}{100}$ = one percent = 1 out of 100 parts = 1%

Look at the chart on the page 175.

Problem 10. *Jim saved \$35. He saved \$10 more than John. How many percent more did Jim save than John?*



$$\$35 - \$10 = \$25 \longrightarrow 100\%$$

$$\downarrow \div 5 \qquad \downarrow \div 5$$

$$\$5 \longrightarrow 20\%$$

$$\downarrow \times 2 \qquad \downarrow \times 2$$

$$\$10 \longrightarrow 40\%$$

Jim saved 40% more than John.

Problem 11. *A store marked up the price of a warm jacket at the beginning of the fall by 30%. For their winter sale, they took 30% off the marked-up price. Is the sale price more, or less, or equal to the original price (before the mark-up)?*

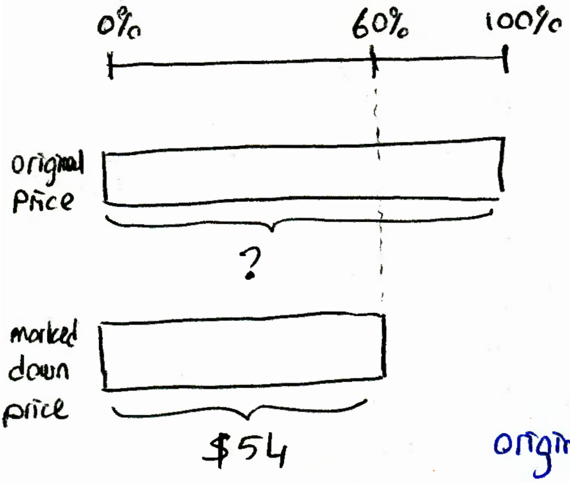
original price

$$\$100 \xrightarrow{\text{30\% increase}} \$130 \xrightarrow{\text{30\% decrease}} \$91$$

sale is \$9 less than the original price

Problem 12. *The price of a shirt was marked down 40% to $54. What was the original price?*

Bar-model solution



60% → $54
↓÷3 ↓÷3
20% → $18
↓×5 ↓×5
100% → $90

original price was $90

Algebraic Solution

Let x be the original price.

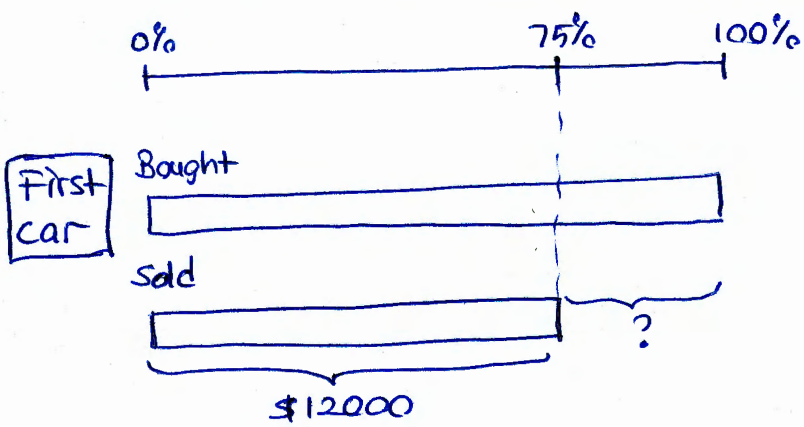
sale price is $54, that is 60% of X

Then $0.6x = 54$

$$x = \frac{54}{0.6} = 90$$

The original price was $90

Problem 13. *A car salesman sold two cars for $12,000 each. The first car was sold at a 25% loss while the second car was sold at a 25% profit. Find the net profit or loss.*

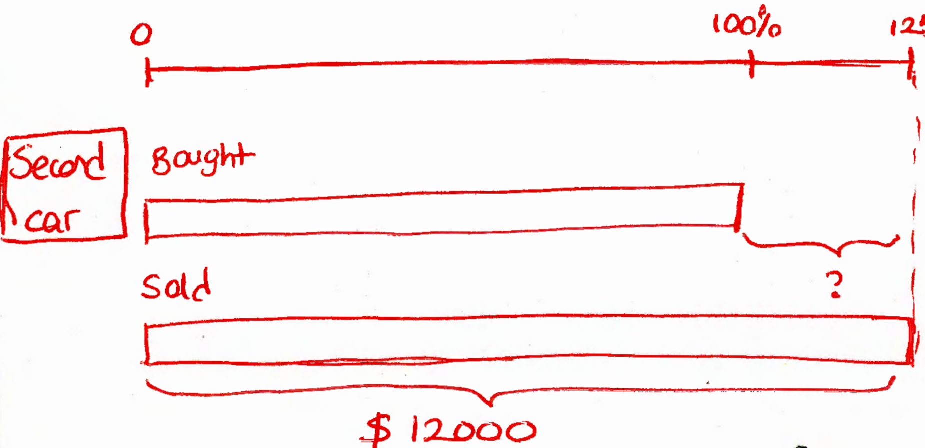


75% → 12,000

÷3 25% → $4,000

x4 100% → $16,000

The lost is $16,000 − $12,000 = $4,000



125% → $12,000

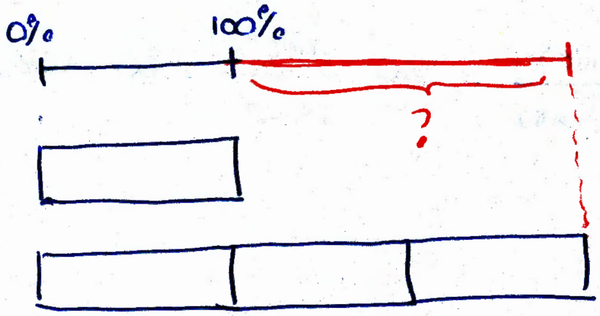
÷5 25% → $ 2,400

x4 100% → $ 9,600

The profit is $12,000 − $9,600 = $2,400

Net loss is $ 4,000 − $ 2,400 = $ 1600

Problem 14. *The value of a stock tripled suddenly. What was the rise as a percentage of the original value?*



1 unit = 100 %

3 units = 300 %

The rise is 300% − 100% = 200%

Problem 15. *The price of a pair of boots was reduced by 20% at the end of the fall season, and the sale price was further reduced by another 50% at the end of the winter. What percent was the overall reduction?*

Assume original price as

$$\$100 \xrightarrow{\text{20\% reduced}} \$80 \xrightarrow{\text{50\% reduced}} \$40$$

$$\underbrace{\$100 \longrightarrow \$40}_{? \,=\, 60\%\ \text{reduction}}$$

Definition 2. *A rate is the quotient of two quantities made with different specified units.*

Problem 16. *A van travels 400 miles in 5 hours. What is the average speed?*

$$\text{Speed} = \frac{400 \text{ miles}}{5 \text{ hours}} = \frac{400}{5} \frac{\text{miles}}{\text{hours}} = 80 \frac{\text{miles}}{\text{hour}} = 80 \text{ miles per hour}$$

Problem 17. *A train travels 315 km at an average speed of 70 km/h. What was the time taken?*

$$70 \frac{\text{km}}{\text{hr}} = \frac{70 \text{ km}}{1 \text{ hr}} = \frac{315 \text{km}}{? \text{ hr}}$$

x4.5

x4.5

? = 4.5 hrs

Problem 18. *Miles is offered a job to paint a house for \$500. He figured it would take him about 25 hours to finish it. He also has another offer to paint a larger house for \$700, and he thinks that would take him 36 hours. What are the rates he is being paid for each job? If he takes both jobs, what is his average pay rate?*

rate for the first job $= \frac{\$500}{25 \text{hrs}} = \frac{500}{25} \frac{\text{dollars}}{\text{hours}} = 20 \frac{\text{dollars}}{\text{hr}} = \$ 20$ per hour

rate for the Second Job $= \frac{\$700}{36 \text{hrs}} = \frac{700}{36} \frac{\text{dollars}}{\text{hrs}} \cong 19.4 \frac{\text{dollars}}{\text{hr}} = \$ 19.4$ per hour

Average pay rate: $\frac{\$ 500+700}{25+36 \text{ hrs}} = 19.67$ dollars per hour

closer to the larger/bigger rate

* Note that average pay rate is not the average of two rates

Problem 19. *Jane is traveling to Toronto from Detroit. While in the US, the speed limit is 70 miles/hour. When she crosses the border, the speed limit is 100 km/hour. Using the approximation that 1 mile ≈ 1.6 km, determine which of the two speed limits is higher, and find the difference.*

$$70 \frac{\cancel{\text{miles}}}{\text{hour}} \cdot \frac{1.6\text{ km}}{1\ \cancel{\text{mile}}} = 112 \frac{\text{km}}{\text{hr}} \longrightarrow \text{the second speed is higher}$$

$112 - 100 = 12 \frac{\text{km}}{\text{hr}}$ is the difference between two speeds

Problem 20. *(Work this problem at home with a calculator.) Michael Phelps can swim 100 meters freestyle in 47.5 seconds. Convert this to miles per hour using the approximation that 1 mile ≈ 1600 meters.*

$$\frac{100\ \cancel{\text{meters}}}{47.5\ \cancel{\text{seconds}}} \cdot \frac{1\text{ mile}}{1600\ \cancel{\text{meters}}} \cdot \frac{60\ \cancel{\text{sec.}}}{1\ \cancel{\text{min.}}} \cdot \frac{60\ \cancel{\text{min}}}{1\text{ hr}} = \frac{100 \cdot 60 \cdot 60}{47.5 \times 1600} \frac{\text{miles}}{\text{hrs}}$$

$$= \frac{60{,}000}{47.5 \times 1600} \frac{\text{miles}}{\text{hrs}}$$

$$= 0.79 \text{ miles per hour}$$

Problem 21. *Al, Bob, and Carl can finish a job (the same job) in 4, 5, and 6 hours respectively. [leave answers in fraction]*

1. *Who is the most efficient?*

Al works at the rate of $\frac{1}{4}$, Bob works at the rate of $\frac{1}{5}$, Carl works at the rate of $\frac{1}{6}$ jobs per hour. Al is the most effcient one.

2. *Who is the least efficient?*

Carl is the least effcient one because she completes least amount of job per hour.

3. *At what rate would each work? Express your answer in job/hr.*

Al: $\frac{1}{4}$ job per hour

Bob: $\frac{1}{5}$ job per hour

Carl: $\frac{1}{6}$ job per hour

4. If Al and Bob work at that job together, how long will it take to finish?

$$\left(\frac{1}{4}+\frac{1}{5}\right)\frac{\text{Jobs}}{\text{hour}} = \frac{9}{20}\ \frac{\text{Jobs}}{\text{hour}} = \frac{9}{20} \text{ Jobs per hour}$$

$$\downarrow$$

$$2\frac{2}{9} = \frac{20}{9} \text{ hours per Job}$$

5. If all three work together how long will it take to finish?

$$\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\frac{\text{Jobs}}{\text{hour}} = \left(\frac{15+12+10}{60}\right)\frac{\text{Jobs}}{\text{hour}}$$

$$= \frac{37}{60} \text{ Jobs per hour}$$

$$\downarrow$$

$$\frac{60}{37} = 1.62 \text{ hours per Job (hours/Job)}$$

6. After they have worked together for an hour, Carl gets a call from home and has to leave. How long will it take Al and Bob to finish the rest of the job?

After an hour, three of them will complete $\frac{37}{60}$ Jobs, and then $\frac{23}{60}$ of the Job will remain

$$\left(\frac{1}{4}+\frac{1}{5}\right)\frac{\text{Jobs}}{\text{hours}} = \frac{9}{20}\ \frac{\text{Jobs}}{\text{hour}} \longrightarrow \frac{\cancel{20}}{9}\ \frac{\text{hours}}{\cancel{\text{Job}}} \cdot \frac{23}{\cancel{60}_{3}}\ \cancel{\text{Job}} = \frac{23}{27} \text{ hours}$$

$\frac{1}{A}+\frac{1}{B}=\frac{1}{3}$ $\frac{1}{A}+\frac{1}{C}=\frac{1}{2}$ $\frac{1}{B}+\frac{1}{C}=\frac{1}{4}$

Problem 22. *Machines A and B take 3 hours to finish a job. Machines A and C take 2 hours to finish the same job. While machines B and C take 4 hours to finish that job.*

1. *Which machine is the most efficient?*

A

2. *Which machine is the least efficient?*

B

3. *At what rate (jobs per hour) would each machine work?*

1st way:

$\frac{1}{A}+\frac{1}{B}-\left(\frac{1}{A}+\frac{1}{C}\right)=\frac{1}{3}-\frac{1}{2}$

$\cancel{\frac{1}{A}}+\frac{1}{B}-\cancel{\frac{1}{A}}-\frac{1}{C}=-\frac{1}{6}$

$\frac{1}{B}-\frac{1}{C}=-\frac{1}{6}$

$\frac{1}{B}+\frac{1}{C}=\frac{1}{4}$

$+$

$\frac{1}{B}-\cancel{\frac{1}{C}}+\frac{1}{B}+\cancel{\frac{1}{C}}=\frac{1}{4}-\frac{1}{6}$

$\frac{2}{B}=\frac{2}{24}$

$\frac{1}{B}=\frac{1}{24}$

$\frac{1}{A}=\frac{1}{3}-\frac{1}{24}=\frac{7}{24}$ Jobs/hr

2nd way:

x: rate of mach. A in J/hr

y: rate of mach. B in J/hr

z: rate of mach. C in J/hr

$\left.\begin{aligned} x+y&=\frac{1}{3}\\ y+z&=\frac{1}{4}\\ x+z&=\frac{1}{2}\end{aligned}\right\}$

$x+y-(y+z)=\frac{1}{3}-\frac{1}{4}$

$x-z=\frac{1}{12}$

$x+z=\frac{1}{2}$

$+$

$2x=\frac{7}{12} \rightarrow x=\frac{7}{24}$

4. *How long does it take Machines A and C (working together) to finish 17 of the above jobs?*

$\frac{1}{A}+\frac{1}{B}=\frac{1}{2}$ Jobs per hour

mach. A & mach. B $\rightarrow$ 2 hours per Job

$\downarrow \times 17$

34 hours for 17 Jobs

OR

$\frac{2 \text{ hours}}{1 \text{ Job}}=\frac{? \text{ hours}}{17 \text{ Jobs}}$ (×17)

$?=34$ hours