Tâtonnement Beyond Constant Elasticity of Substitution

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The Fisher Market Model

Model

- m goods.
  - Unit supply
- n buyers. For each buyer i:
  - A utility function $u_i: \mathbb{R}^m \to \mathbb{R}$
  - A budget $b_i \in \mathbb{R}_+$
- We consider **continuous, concave, and homogeneous** utility functions
  - Homogeneous: $u_i(\lambda \boldsymbol{x}) = \lambda u_i(\boldsymbol{x})$

An **outcome** of a Fisher market:

- Allocations of goods to buyers
- Prices for goods

An allocation and prices are a **Competitive (or Walrasian) equilibrium** if:

1) The buyers' allocations are utility maximizing constrained by their budget

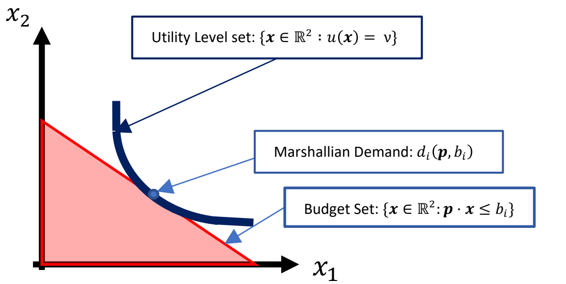
2) For each good j, either:

- If price > 0, Demand = Supply
- If price = 0, Demand $\leq$ Supply

A Consumer Theory Duality Primer

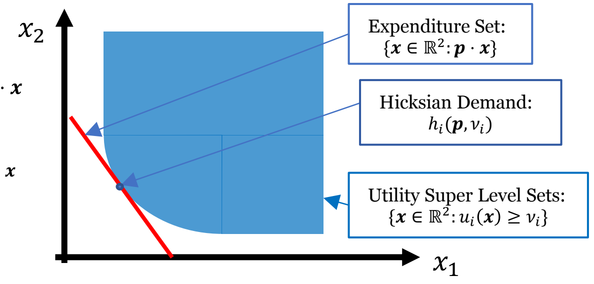
Utility Maximization:

- Indirect Utility:
  - $v_i(\boldsymbol{p}, b_i) = \max_{\boldsymbol{x}:\boldsymbol{p}\cdot\boldsymbol{x}\leq b_i} u_i(\boldsymbol{x})$
- Marshallian Demand:
  - $d_i(\boldsymbol{p}, b_i) = \operatorname{argmax}_{\boldsymbol{x}:\boldsymbol{p}\cdot\boldsymbol{x}\leq b_i} u_i(\boldsymbol{x})$



Expenditure Minimization:

- Expenditure:
  - $e_i(\boldsymbol{p}, v_i) = \min_{\boldsymbol{x}:u_i(\boldsymbol{x})\geq v_i} \boldsymbol{p}\cdot\boldsymbol{x}$
- Hicksian Demand
  - $h_i(\boldsymbol{p}, v_i) = \operatorname{argmin}_{\boldsymbol{x}:u_i(\boldsymbol{x})\geq v_i} \boldsymbol{p}\cdot\boldsymbol{x}$



A Natural Price Adjustment Process: Tâtonnement

- Define the **excess demand** $z(\boldsymbol{p}, \boldsymbol{b})$ for good j at prices $\boldsymbol{p}$ and budgets $\boldsymbol{b}$:

$$z(\boldsymbol{p}, \boldsymbol{b}) = \sum_i d_i(p, b_i) - \mathbf{1}_m$$

- The discrete tâtonnement process is defined as:

$$\boldsymbol{p}(t+1) = \boldsymbol{p}(t) + z(\boldsymbol{p}(t), \boldsymbol{b})$$

Interpretation: If demand > supply, increase price
If demand < supply, decrease price

Gradient Descent as Tâtonnement

- (Sub)gradient of our program at a price $\boldsymbol{p}$ is equal to negative excess demand!

$$\partial_{\boldsymbol{p}} \left(\underbrace{\sum_j p_j - \sum_i b_i \log \frac{\partial e_i(\boldsymbol{p}, v_i)}{\partial v_i}}_{\text{Our objective Function}} \right) = \underbrace{-z(\boldsymbol{p}, \boldsymbol{b})}_{\text{Negative Excess Demand}}$$

- (Sub)gradient descent on our convex program is then equivalent to:

$$\underbrace{\boldsymbol{p}(t+1)}_{\text{Next Price}} = \underbrace{\boldsymbol{p}(t)}_{\text{Previous Price}} + \underbrace{z(\boldsymbol{p}(t), \boldsymbol{b})}_{\text{Negative Subgradient}}$$

⇔ Excess demand!

Results

Result 1: We generalize the Eisenberg-Gale program's dual and propose a convex program to compute and characterize equilibrium prices of Fisher markets via expenditure functions

- Through expenditure functions, constraints are abstracted out and program is unconstrained!
- Eisenberg-Gale's and Shymrev's program are special cases!
- **Interpretation**: Equilibrium prices are those that minimize distance between auctioneer's surplus and consumer surplus!

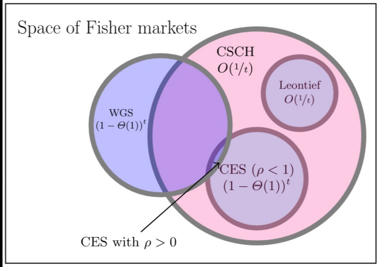
$$\min_{\boldsymbol{p}} \underbrace{\sum_{j\in[m]} p_j}_{\text{Seller's Surplus}} - \underbrace{\sum_{i\in[n]} b_i \log \left(\overbrace{\frac{\partial e_i(\boldsymbol{p}, v_i)}{\partial v_i}}^{\text{Marginal cost of utility}} \right)}_{\text{Buyers' Surplus}}$$

Result 2: We extend the $O\left(\frac{1}{t}\right)$ convergence rate of tâtonnement to a more general class of Fisher Markets, by using the equivalence between tâtonnement and gradient descent on our program

1. We show that (generalized) gradient descent on our convex program is equivalent to the tâtonnement process

2. We use results on the convergence of generalized gradient descent to prove the convergence rate of tâtonnement in continuous, strictly concave, and homogeneous (CSCH) Fisher markets

Market Type	Convergence Rate	Authors, Year
Fisher CES (excluding Linear)	$(1-\Theta(1))^t$	Cheung et al. 2020
Fisher Leontief	$O\left(\frac{1}{t}\right), \Omega\left(\frac{1}{t^2}\right)$	Cheung et al. 2020
Fisher Linear	$(1-\Theta(1))^t$ (Under Large Market Assumption)	Cole and Tao 2019
Fisher WGS	$(1-\Theta(1))^t$	Cole and Fleischer 2008
Fisher CSCH	$O\left(\frac{1}{t}\right)$	Our Results



A Consumer Theory Duality Primer

- **Utility Maximization:**
  - Indirect Utility:
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  - Marshallian Demand:
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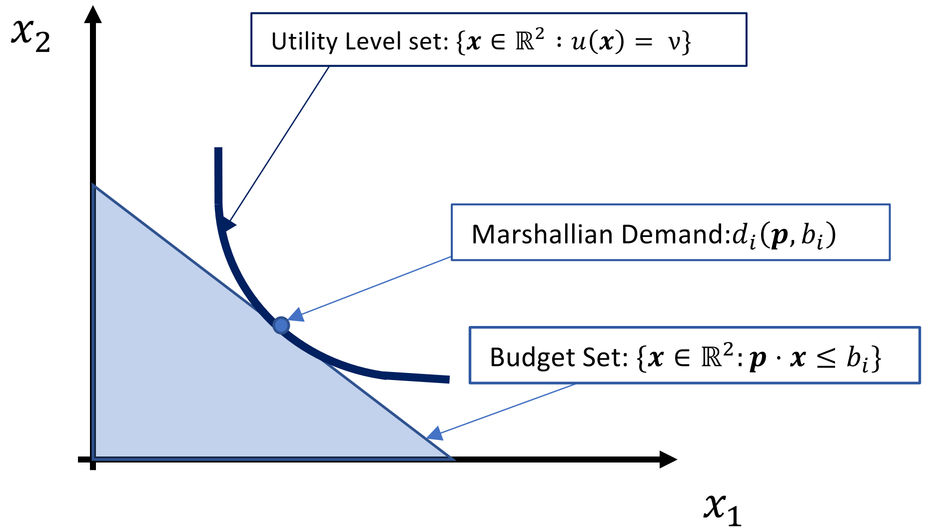

- **Expenditure Minimization:**
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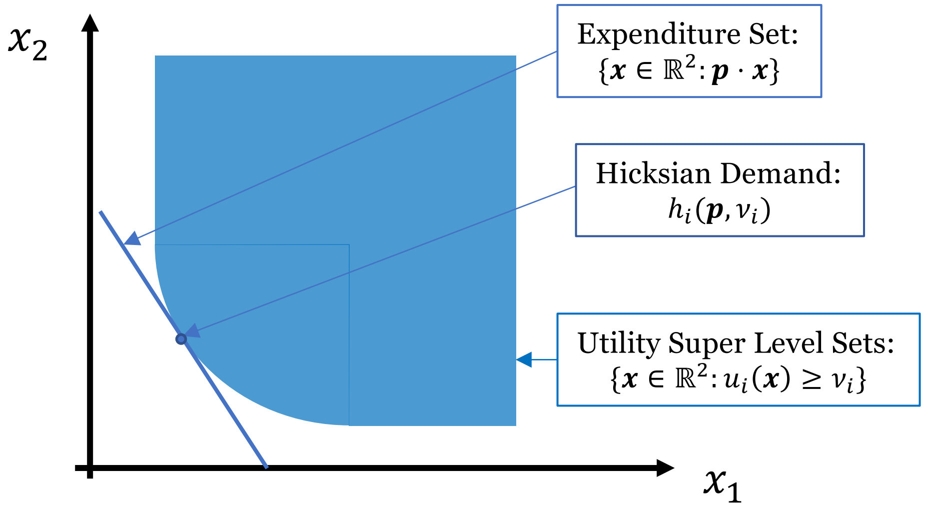

Our Convex Program: Equilibrium Prices through Expenditure functions

Marginal cost of utility

$$\min_{\boldsymbol{p}} \underbrace{\sum_{j \in [m]} p_j}_{\text{Seller's Surplus}} - \underbrace{\sum_{i \in [n]} b_i \log \left(\overbrace{\frac{\partial e_i(\boldsymbol{p}, v_i)}{\partial v_i}}^{\text{Marginal cost of utility}} \right)}_{\text{Buyers' Surplus}}$$

Seller's Surplus

Buyers' Surplus

- **Result**: We generalize the Eisenberg-Gale program's dual and propose a convex program to compute and characterize equilibrium prices of Fisher markets via expenditure functions
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$$\underbrace{z(\boldsymbol{p}, \boldsymbol{b})}_{\text{Excess Demand}} = \underbrace{\sum_{\boldsymbol{i}} d_i(p, b_i)}_{\text{Demand}} - \underbrace{\boldsymbol{1}_{\boldsymbol{m}}}_{\text{Supply}}$$

- The discrete tâtonnement process is defined as:

$$\underbrace{\boldsymbol{p}(t+1)}_{\text{Next Price}} = \underbrace{\boldsymbol{p}(t)}_{\text{Previous Price}} + \underbrace{z(\boldsymbol{p}(t), \boldsymbol{b})}_{\text{Excess Demand}}$$

Interpretation: If demand > supply increase price
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A New Stability Result

- **Result:** We extend the $O\left(\frac{1}{t}\right)$ convergence rate of tâtonnement to a more general class of Fisher Markets, by using the equivalence between tâtonnement and gradient descent on our program

1. We show that the (generalized) gradient descent on our convex program is equivalent to the tâtonnement process
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Summary of Tâtonnement Convergence Results For Fisher Markets

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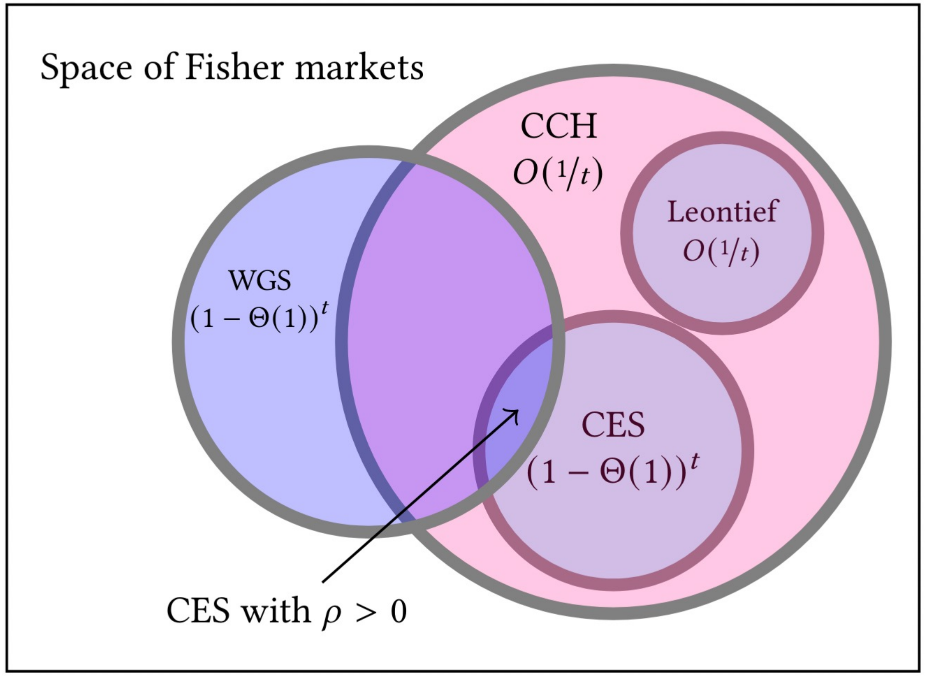

Previous convergence results in blue
our result in red